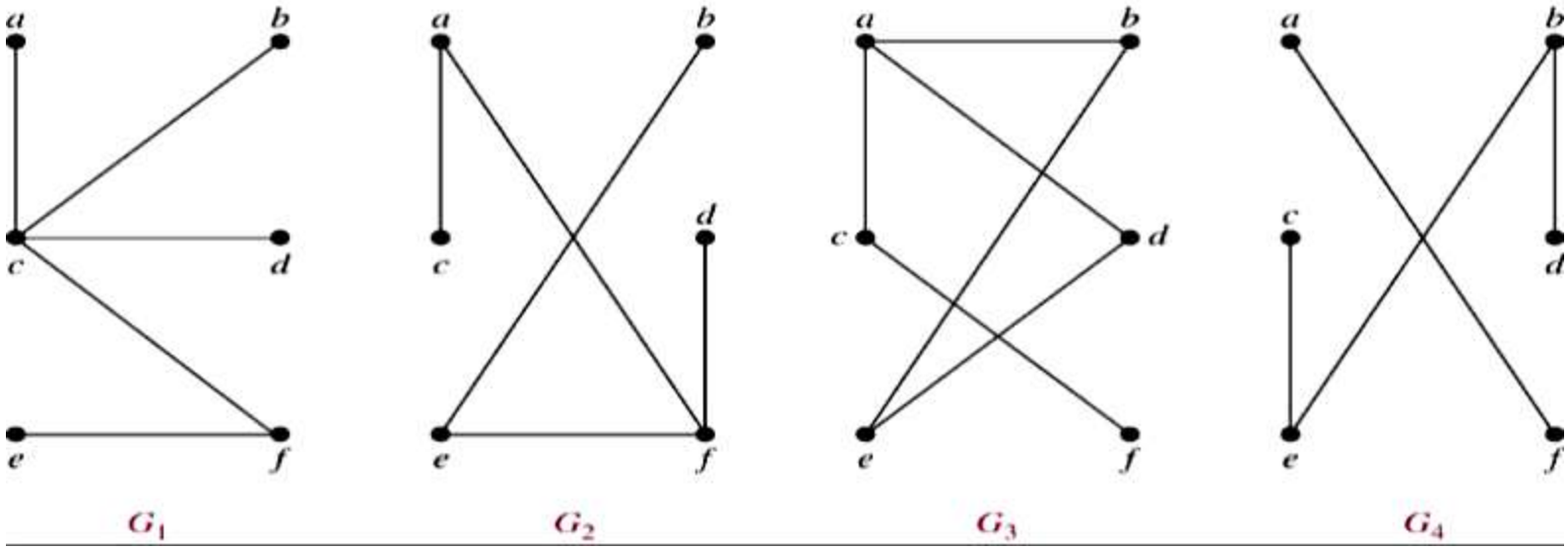
Outline

- 10.1 Introduction to Trees
- 10.2 Applications of Trees
- 10.3 Tree Traversal
- 10.4 Spanning Trees
- 10.5 Minimal Spanning Trees

10.1 Introduction to Trees

Def 1 A tree is a connected undirected graph with no simple circuits.

Example 1. Which of the graphs are trees?



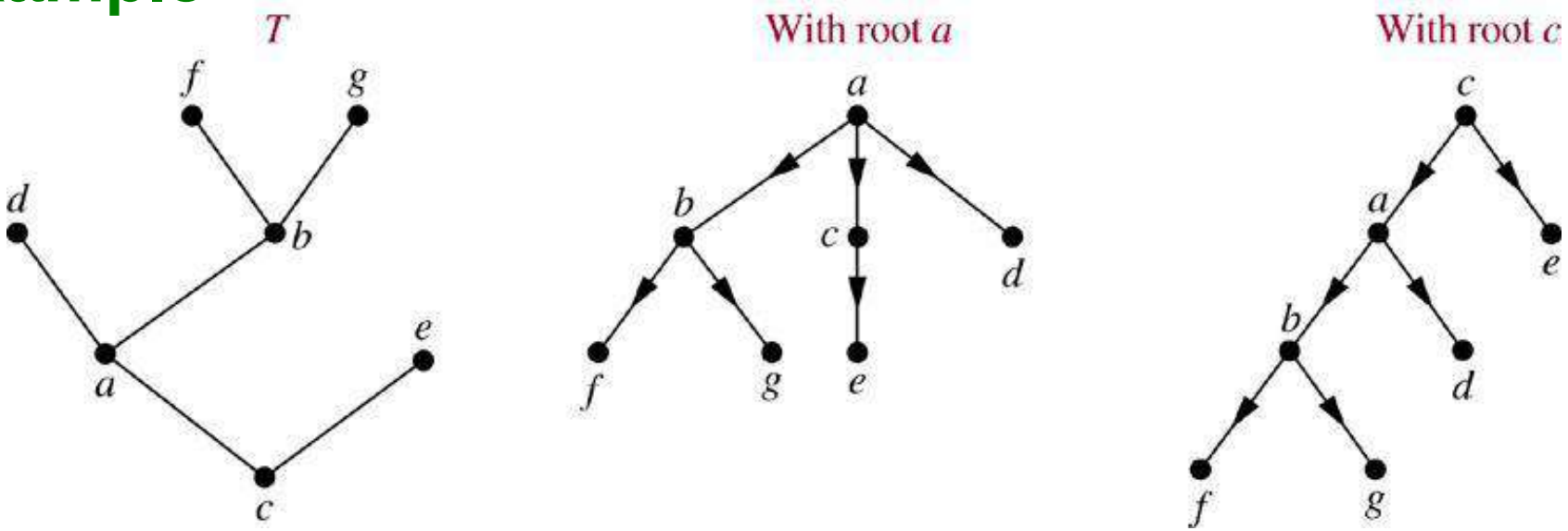
Sol: G_1, G_2

Note. 若拿掉connected的條件，就變成 forest

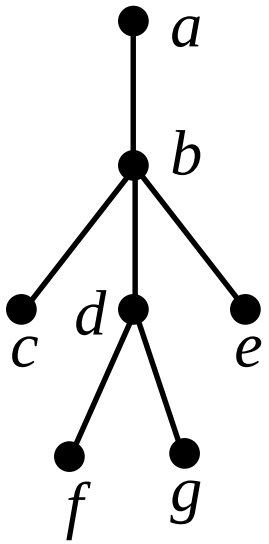
Thm 1. Any undirected graph is a tree if and only if there is a unique simple path between any two of its vertices.

Def 2. A rooted tree is a tree in which one vertex has been designed as the root and every edge is directed away from the root. (箭頭可消掉)

Example



Def:



a is the parent of b, b is the child of a,

c, d, e are siblings,

a, b, d are ancestors of f

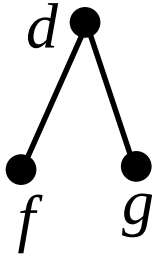
c, d, e, f, g are descendants of b

c, e, f, g are leaves of the tree (deg=1)

a, b, d are internal vertices of the tree

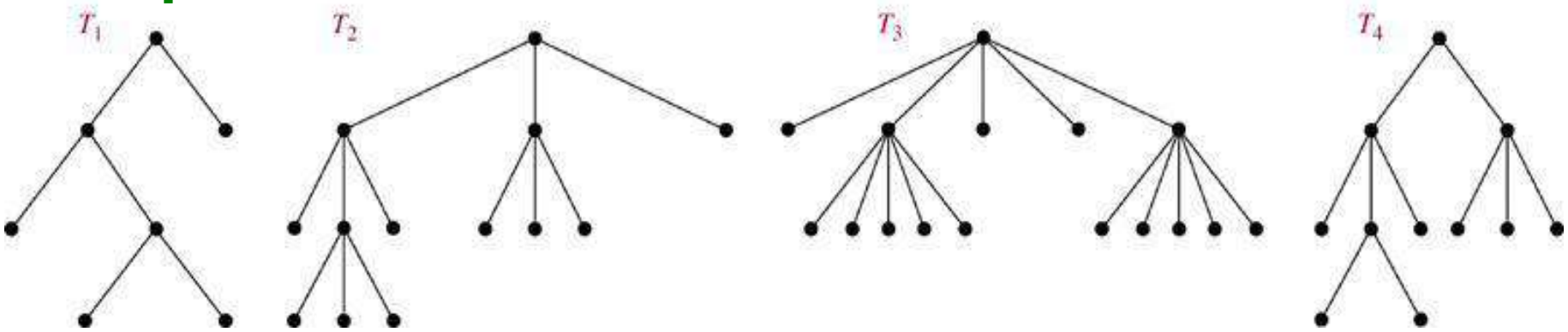
(at least one child)

subtree with d as its root:



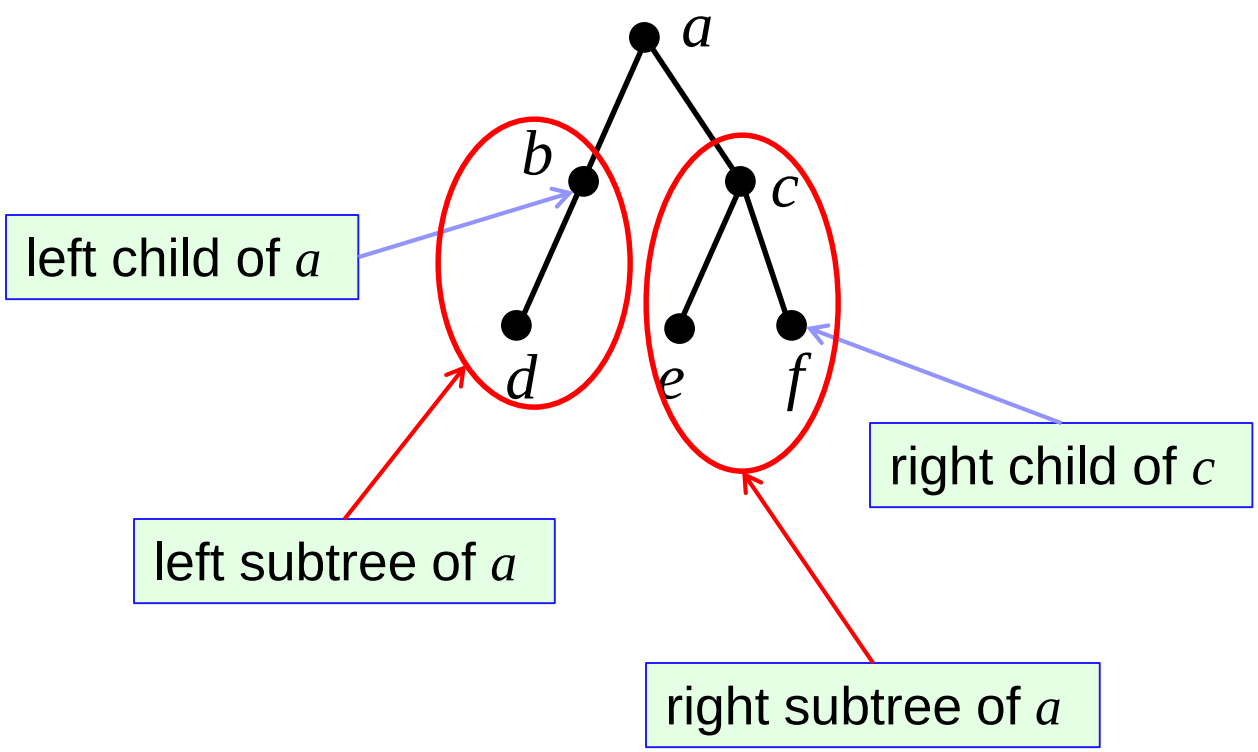
Def 3 A rooted tree is called an m-ary tree if every internal vetex has no more than m children. The tree is called a full m-ary tree if every internal vertex has exactly m children. An m-ary tree with $m=2$ is called a binary tree.

Example 3



full binary tree full 3-ary tree full 5-ary tree not full 3-ary tree

Def:


a
b
c
d
e
f
left child of a
right child of c
left subtree of a
right subtree of a

Properties of Trees

Thm 2. A tree with n vertices has $n-1$ edges.

Pf. (by induction on n)

$n = 1$: K_1 is the only tree of order 1, $|E(K_1)| = 0$. ok!

Assume the result is true for every trees of order $n = k$.

Let T be a tree of order $n = k+1$, v be a leaf of T, and w be the parent of v.

Let T' be the tree $T - \{v\}$.

$\therefore |V(T')| = k$, and $|E(T')| = k-1$ by the induction hypothesis.

$\Rightarrow |E(T)| = k$

By induction, the result is true for all trees. #

Thm 3. A full m-ary tree with i internal vertices contains $n = mi + 1$ vertices.

Pf. Every vertex, except the root, is the child of an internal vertex.

Each internal vertex has m children.

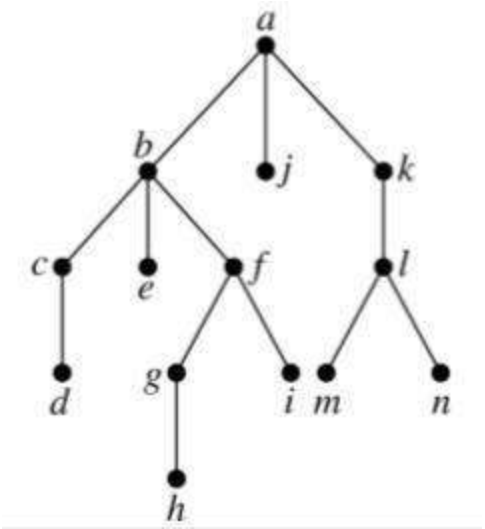
$\Rightarrow$ there are $mi + 1$ vertices in the tree

Exercise: 19

Cor. A full m-ary tree with n vertices contains $(n-1)/m$ internal vertices, and hence $n - (n-1)/m = ((m-1)n+1)/m$ leaves.

Def: The level of a vertex v in a rooted tree is the length of the unique path from the root to this vertex.
The level of the root is defined to be zero.
The height of a rooted tree is the maximum of the levels of vertices.

Example 10.



level

0

1

2

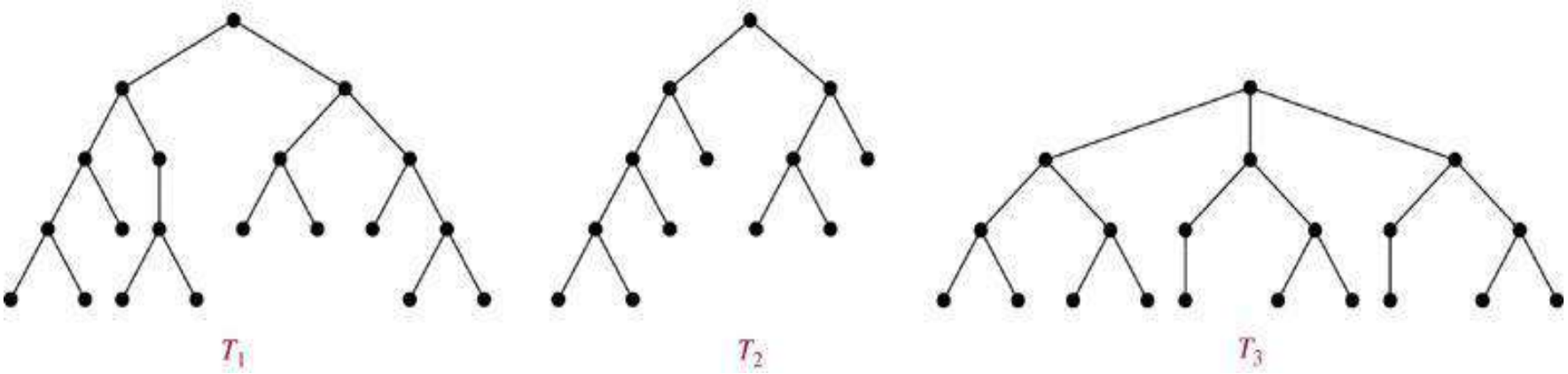
3

4

height = 4

Def: A rooted m-ary tree of height h is balanced if all leaves are at levels h or $h-1$.

Example 11 Which of the rooted trees shown below are balanced?



Sol. T_1, T_3

Thm 5. There are at most m^h leaves in an m-ary tree of height h.

Def: A complete m-ary tree is a full m-ary tree, where every leaf is at the same level.

Ex 28 How many vertices and how many leaves does a complete m-ary tree of height h have?

Sol.

# of vertices = $1+m+m^2+\ldots+m^h = (m^{h+1}-1)/(m-1)$

# of leaves = m^h

10.2 Applications of Trees

Binary Search Trees

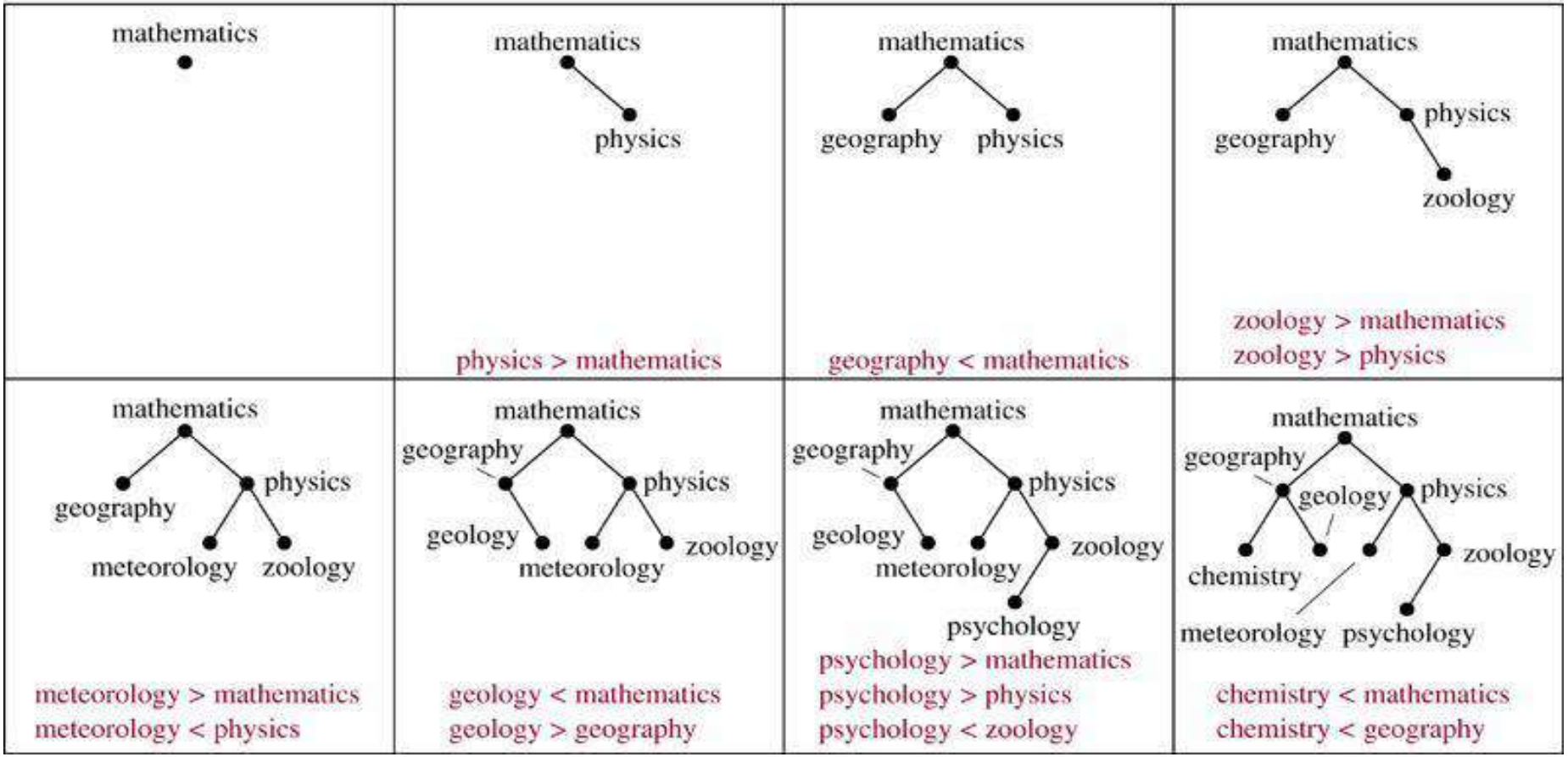
Goal: Implement a searching algorithm that finds items efficiently when the items are totally ordered.

Binary Search Tree: Binary tree + each child of a vertex is designed as a right or left child, and each vertex v is labeled with a key $label(v)$, which is one of the items.

Note: $label(v) > label(w)$ if w is in the left subtree of v and $label(v) < label(w)$ if w is in the right subtree of v

Example 1 Form a binary search tree for the words *mathematics*, *physics*, *geography*, *zoology*, *meteorology*, *geology*, *psychology*, and *chemistry* (using alphabetical order).

Sol.



Algorithm 1 (Locating and Adding Items to a Binary Search Tree.)

Procedure *insertion*(*T*: binary search tree, *x*: item)
v := root of *T*
{a vertex not present in *T* has the value *null*}
while *v* ≠ *null* and *label*(*v*) ≠ *x*
begin
    if *x* < *label*(*v*) **then**
        if left child of *v* ≠ *null* **then** *v*:=left child of *v*
        else add *new vertex* as a left child of *v* and set *v* := *null*
    else
        if right child of *v* ≠ *null* **then** *v*:= right child of *v*
        else add *new vertex* as a right child of *v* and set *v* := *null*
end
if root of *T* = *null* **then** add a vertex *v* to the tree and label it with *x*
else if *v* is null or *label*(*v*) ≠ *x* **then** label new vertex with *x* and
                                                let *v* be this new vertex
{*v* = location of *x*}

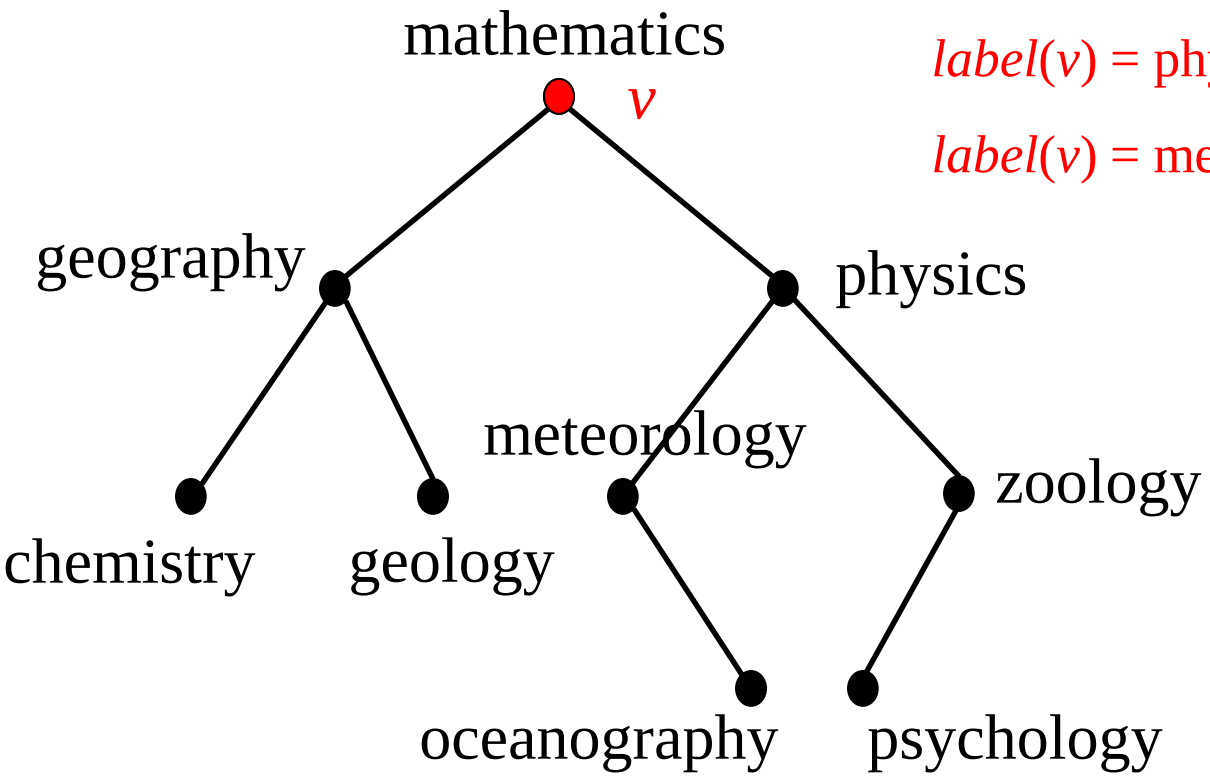
Example 2 Use Algorithm 1 to insert the word *oceanography* into the binary search tree in Example 1.

Sol.

label(*v*) = mathematics < *oceanography*

label(*v*) = physics > *oceanography*

label(*v*) = meteorology < *oceanography*



Exercise: 1,3

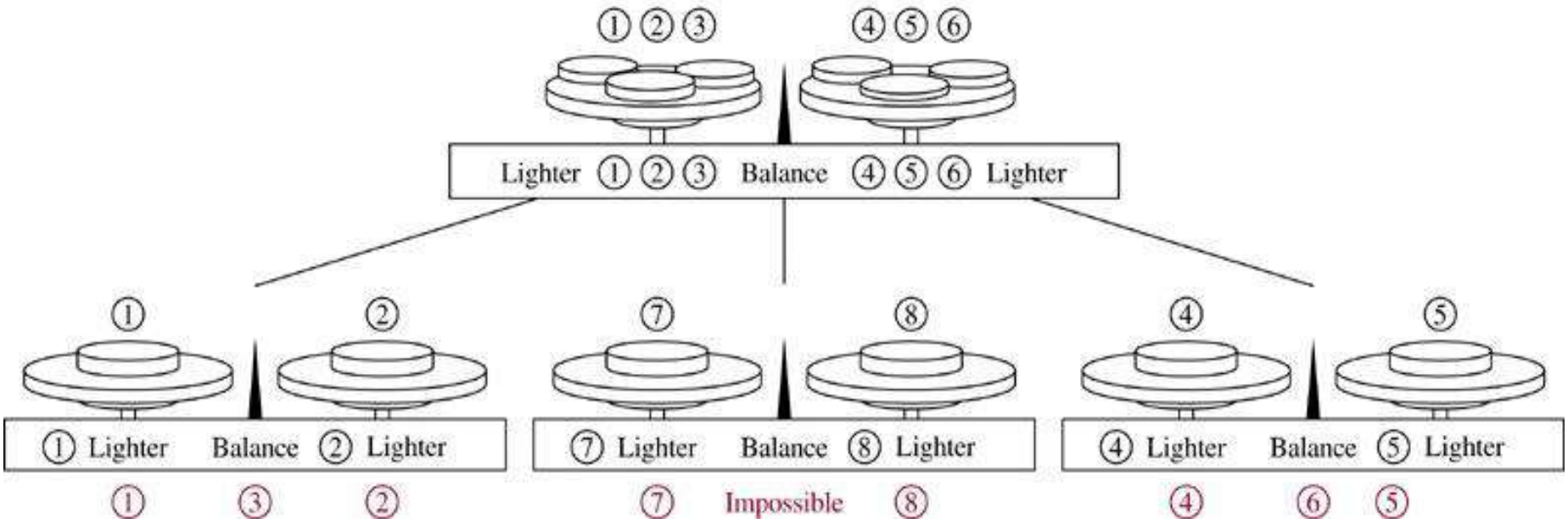
Decision Trees

A rooted tree in which each internal vertex corresponds to a decision, with a subtree at these vertices for each possible outcome of the decision, is called a decision tree.

Example 3 Suppose there are seven coins, all with the same weight, and a counterfeit (偽造) coin that weights less than the others. How many weighings (秤重) are necessary using a balance scale (秤) to determine which of the eight coins is the counterfeit one? Give an algorithm for finding this counterfeit coin.

Sol. 秤重時，可能左重、右重或平衡 ⇒ 3-ary tree

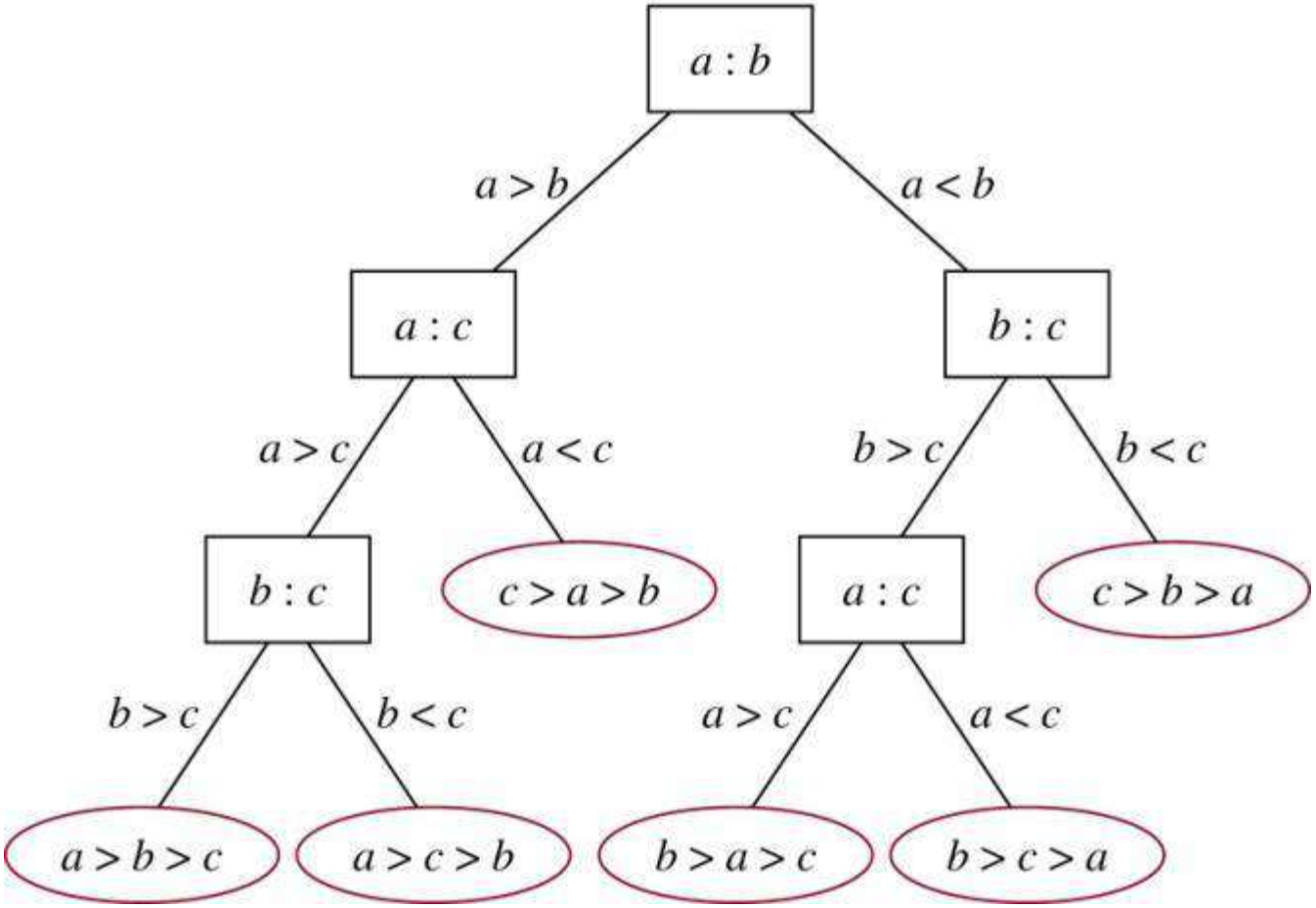
Need 8 leaves ⇒ 至少需秤重兩次



Exercise: 7

Example 4 A decision tree that orders the elements of the list a, b, c.

Sol.



Prefix Codes

Problem: Using bit strings to encode the letter of the English alphabet (不分大小寫)

⇒ each letter needs a bit string of length 5 (因 $2^4 < 26 < 2^5$)

⇒ Is it possible to find a coding scheme of these letter such that when data are coded, fewer bits are used?

⇒ Encode letters using varying numbers of bits.

⇒ Some methods must be used to determine where the bits for each character start and end.

⇒ Prefix codes: Codes with the property that the bit string for a letter never occurs as the first part of the bit string for another letter.

Example: (not prefix code)

e : 0, a : 1, t : 01

The string 0101 could correspond to *eat*, *tea*, *eaea*, or *tt*.

Example: (prefix code)

e : 0, a : 10, t : 11

The string 10110 is the encoding of *ate*.

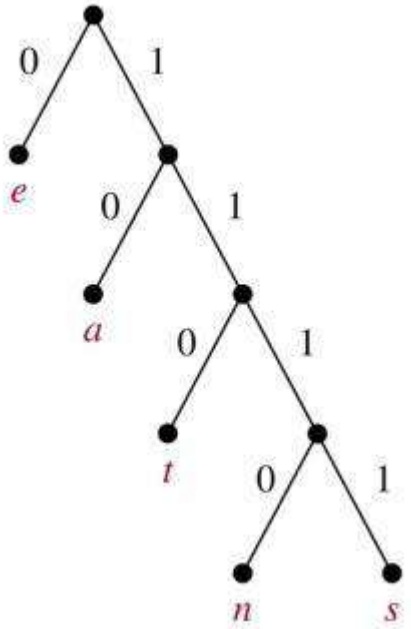
A prefix code can be represented using a binary tree.

character: the label of the leaf

edge label: left child $\rightarrow$ 0, right child $\rightarrow$ 1

The bit string used to encode a character is the sequence of labels of the edges in the unique path from the root to the leaf that has this character as its label.

Example:



encode

e : 0

a : 10

t : 110

n : 1110

s : 1111

從root走起，到leaf為止，重複

decode

11111011100

s a n e

$\Rightarrow$ *sane*

Exercise: 22

Huffman Coding (data compression重要工具)

Input the frequencies of symbols in a string and output a prefix code that encodes the string using the fewest possible bits, among all possible binary prefix codes for these symbols.

一開始有很多孤立點，label就是各個symbol,
將最少使用的兩個symbol結合成一個subtree,
重複此一概念,
將最少使用的兩個subtree結合成一個subtree，…

Algorithm 2 (Huffman Coding)

Procedure *Huffman*(*C*: symbols a_i with frequencies w_i, $i = 1, \ldots, n$)

F := forest of *n* rooted trees, each consisting of the single vertex a_i and assigned weighted w_i

while *F* is not a tree

begin

Replace the rooted trees *T* and *T'* of least weights from *F* with $w(T) \geq w(T')$ with a tree having a new root that has *T* as its left subtree and *T'* as its right subtree. Label the new edge to *T* with 0 and the new edge to *T'* with 1.

Assign $w(T)+w(T')$ as the weight of the new tree.

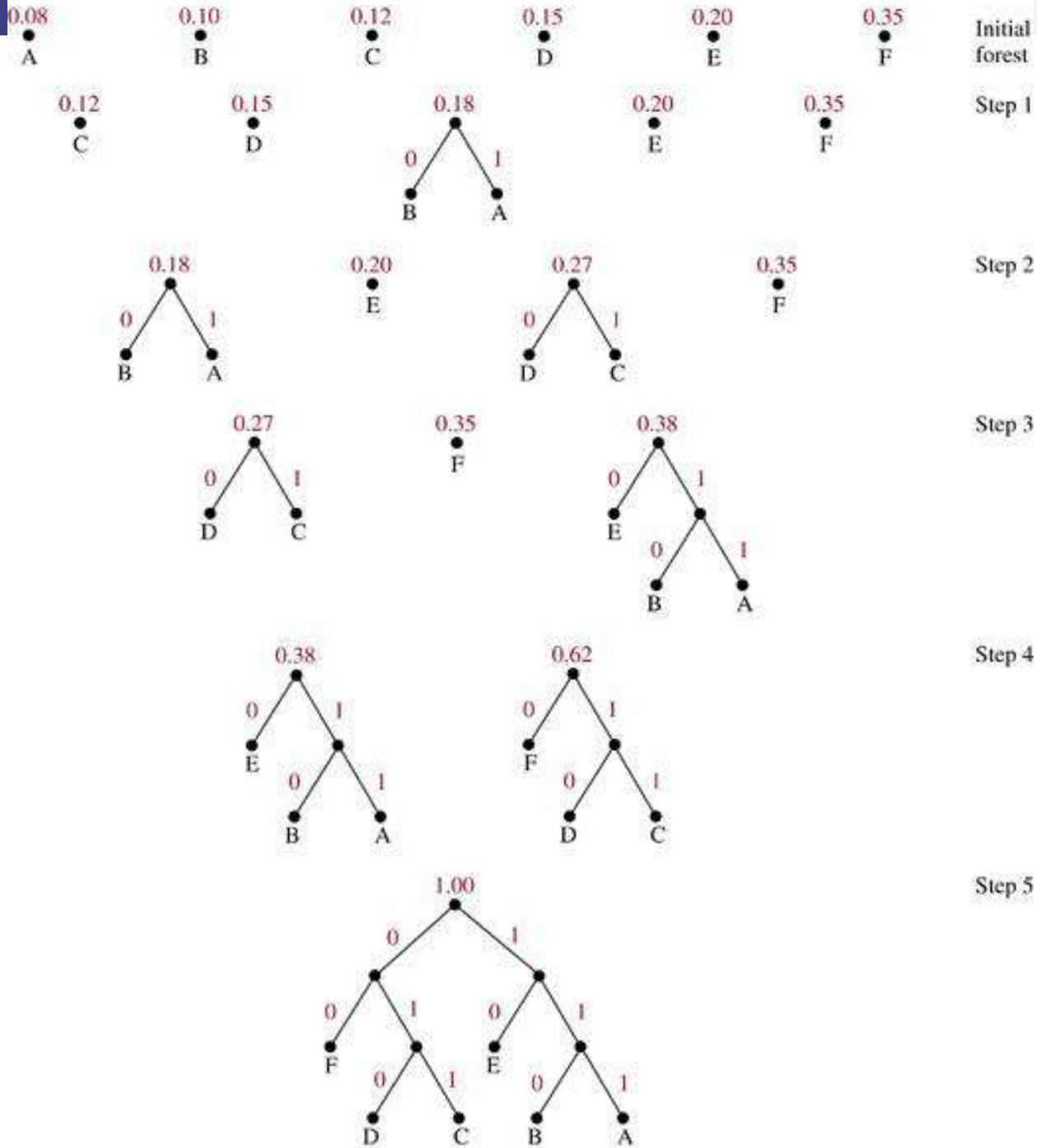
end

Example 5 Use Huffman coding to encode the following symbols with the frequencies listed:
A: 0.08, B: 0.10, C: 0.12, D: 0.15, E: 0.20, F: 0.35.
What is the average number of bits used to encode a character?

Sol:

1. 下頁圖
2. The average number of bits is:
   每個symbol 長度× 頻率 的加總
   $= 3\times0.08+ 3\times0.10+ 3\times0.12+ 3\times0.15+2\times0.20+2\times0.35$
   $=2.45$

Exercise : 23


0.08
A
0.10
B
0.12
C
0.15
D
0.20
E
0.35
F
Initial forest
0.12
C
0.15
D
0.18
0
1
B
A
0.20
E
0.35
F
Step 1
0.18
0
1
B
A
0.20
E
0.27
0
1
D
C
0.35
F
Step 2
0.27
0
1
D
C
0.35
F
0.38
0
1
E
0
1
B
A
Step 3
0.38
0
1
E
0
1
B
A
0.62
0
1
F
0
1
D
C
Step 4
1.00
0
1
0
1
F
0
1
E
0
1
D
C
B
A
Step 5

10.3 Tree Traversal

We need procedures for visiting each vertex of an ordered rooted tree to access data.

Universal Address Systems

Label vertices:

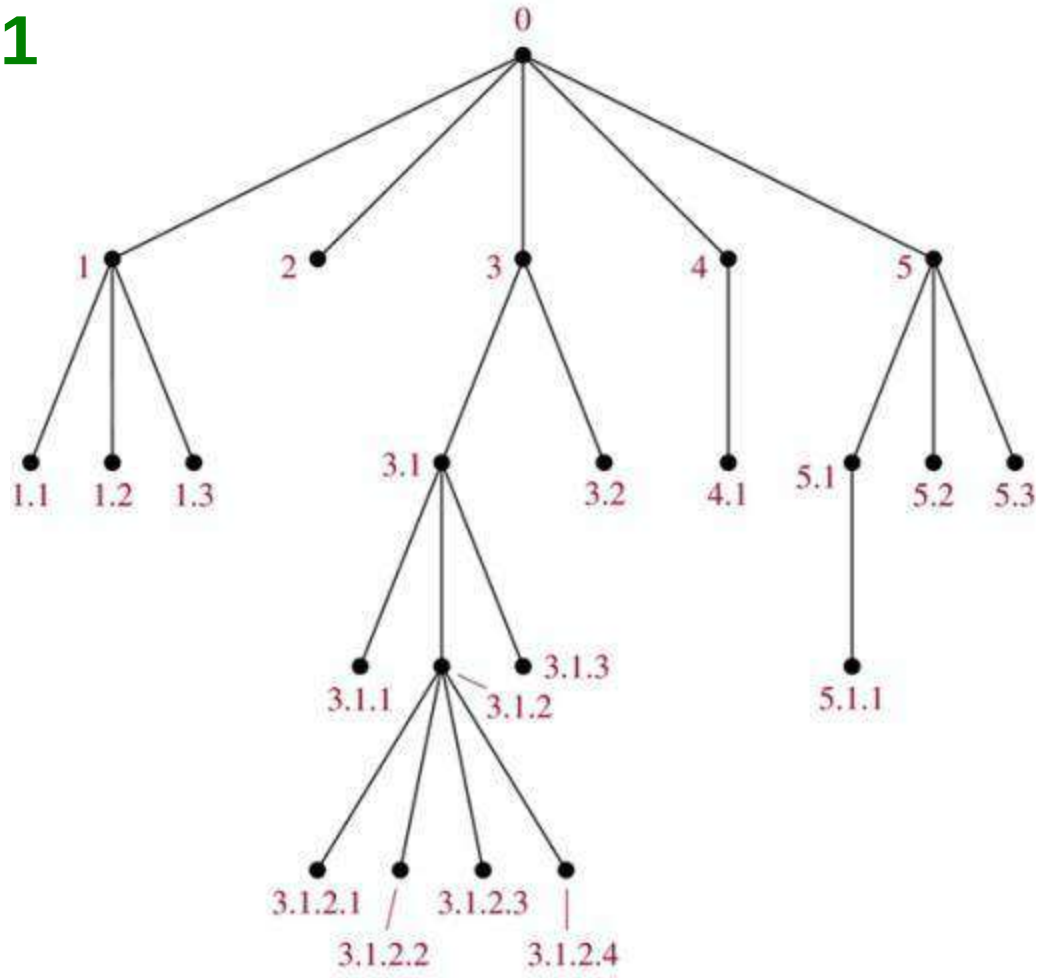
1.root $\rightarrow$ 0, its k children $\rightarrow$ $1, 2, \ldots, k$ (from left to right)

2.For each vertex v at level n with label A,
its r children $\rightarrow$ $A.1, A.2, \ldots, A.r$ (from left to right).

We can totally order the vertices using the lexicographic ordering of their labels in the universal address system.

$x_1.x_2.\ldots.x_n < y_1.y_2.\ldots.y_m$
if there is an i, $0 \leq i \leq n$, with $x_1=y_1$, $x_2=y_2$, …, $x_{i-1}=y_{i-1}$, and $x_i<y_i$;
or if $n<m$ and $x_i=y_i$ for $i=1, 2, \ldots, n$.

Example 1



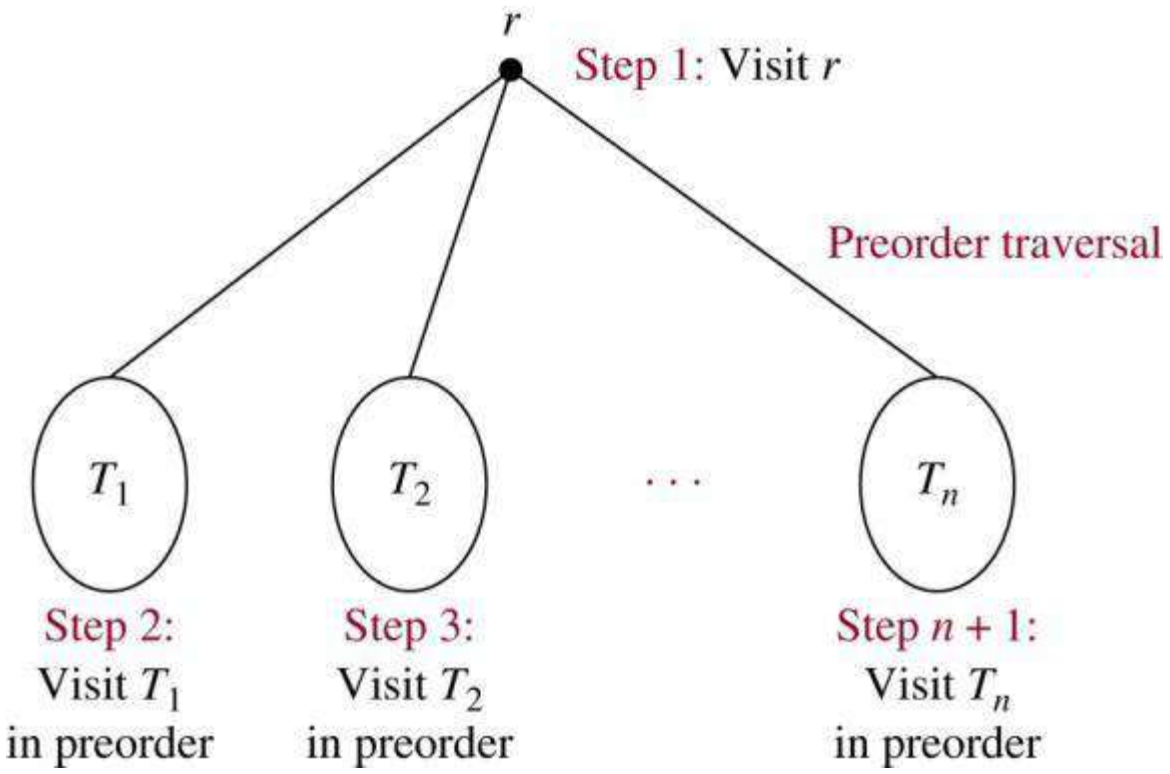
The lexicographic ordering is:

0 < 1<1.1 < 1.2 < 1.3 < 2 < 3 < 3.1 < 3.1.1 < 3.1.2 < 3.1.2.1 < 3.1.2.2 < 3.1.2.3 < 3.1.2.4 < 3.1.3 < 3.2 < 4 < 4.1 < 5 < 5.1 < 5.1.1 < 5.2 < 5.3

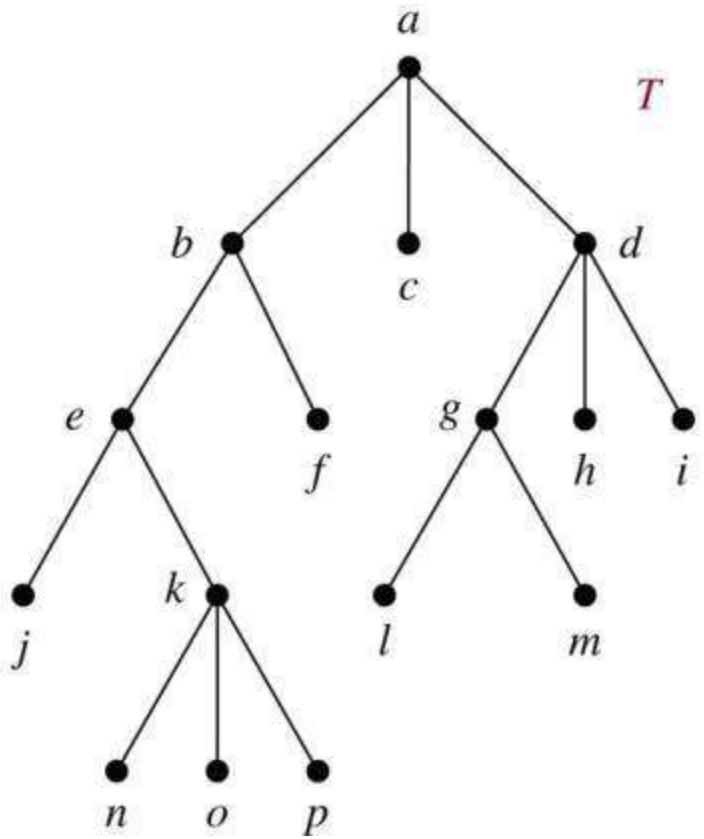
Exercise : 2

Traversal Algorithms

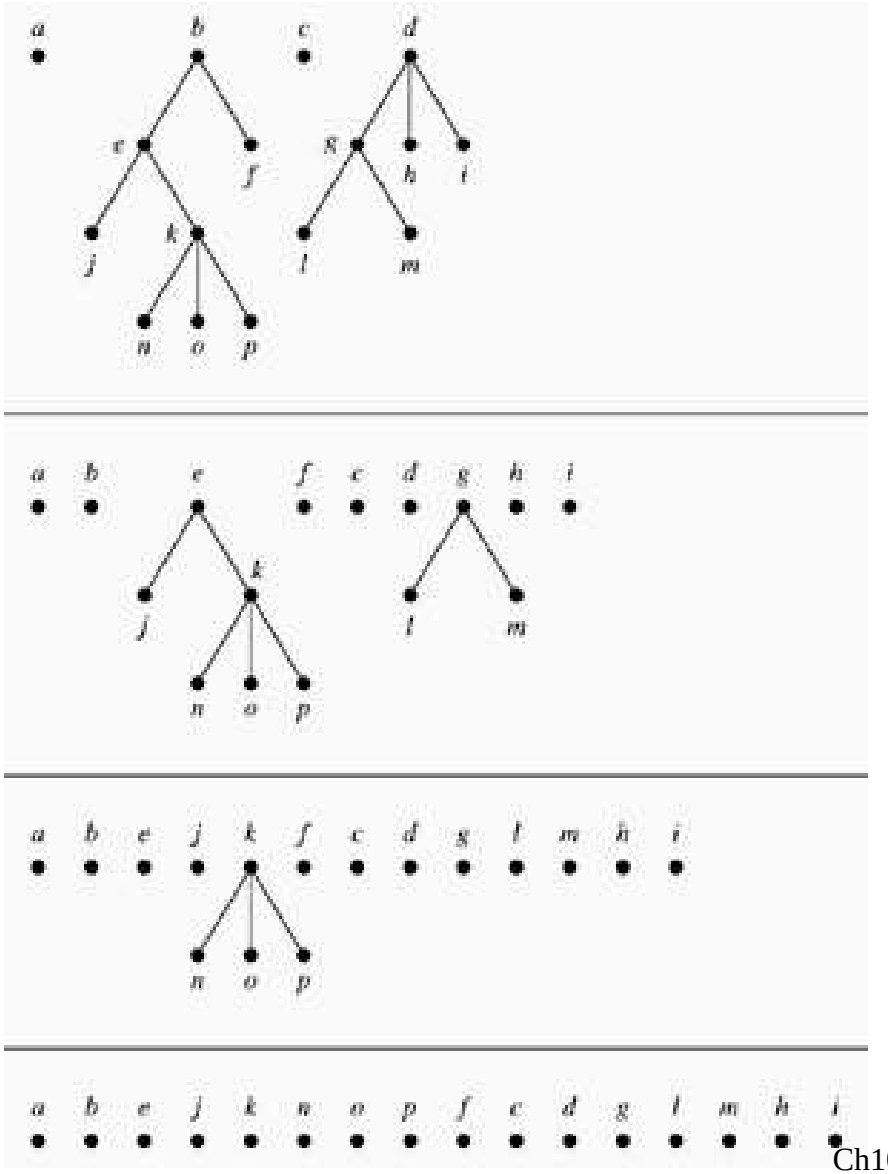
Preorder traversal (前序)



Example 2. In which order does a preorder traversal visit the vertices in the ordered rooted tree T shown below?



Sol:

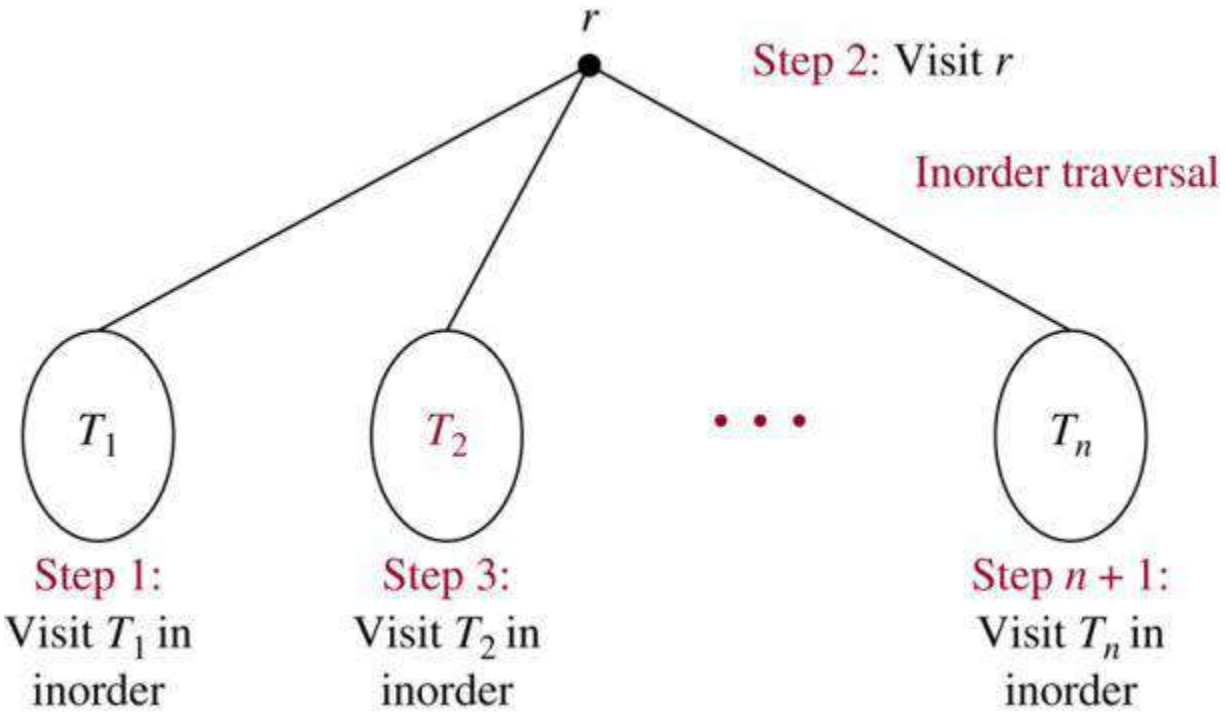


Algorithm 1 (Preorder Traversal)

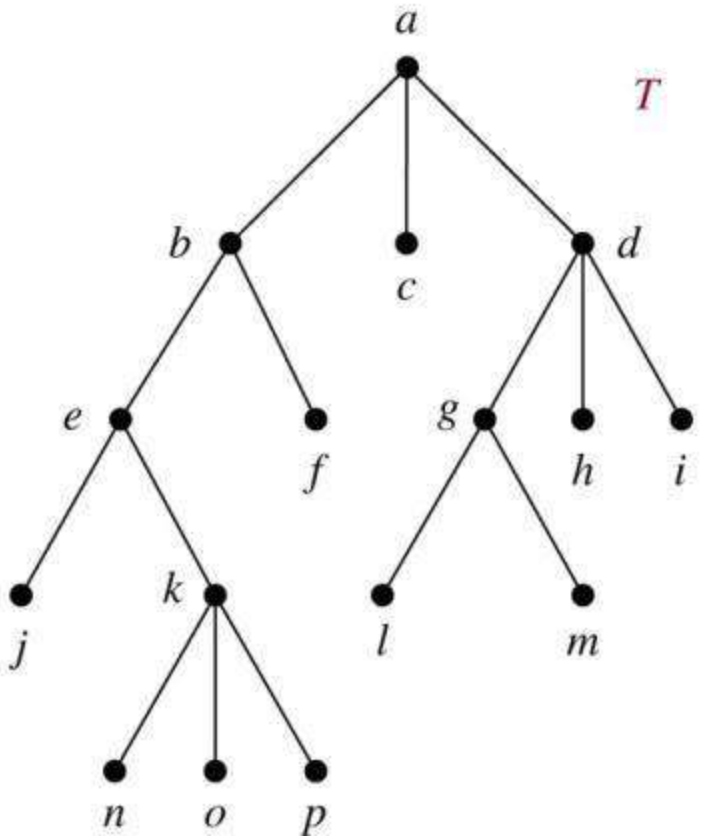
Procedure *preorder*(*T*: ordered rooted tree)
r := root of *T*
list *r*
for each child *c* of *r* from left to right
begin
    T(*c*) := subtree with *c* as its root
    preorder(*T*(*c*))
end

Exercise : 8

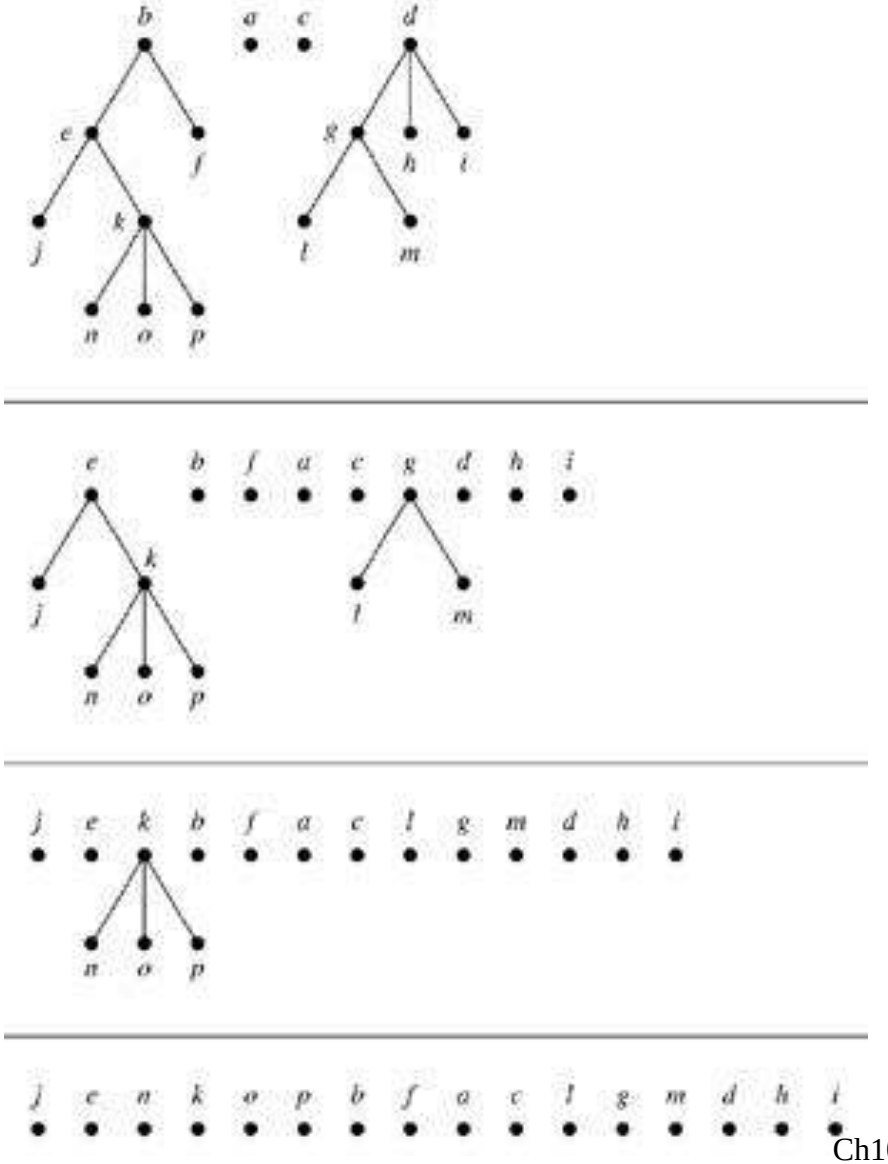
Inorder traversal(中序)



Example 3. In which order does a preorder traversal visit the vertices in the ordered rooted tree T shown below?



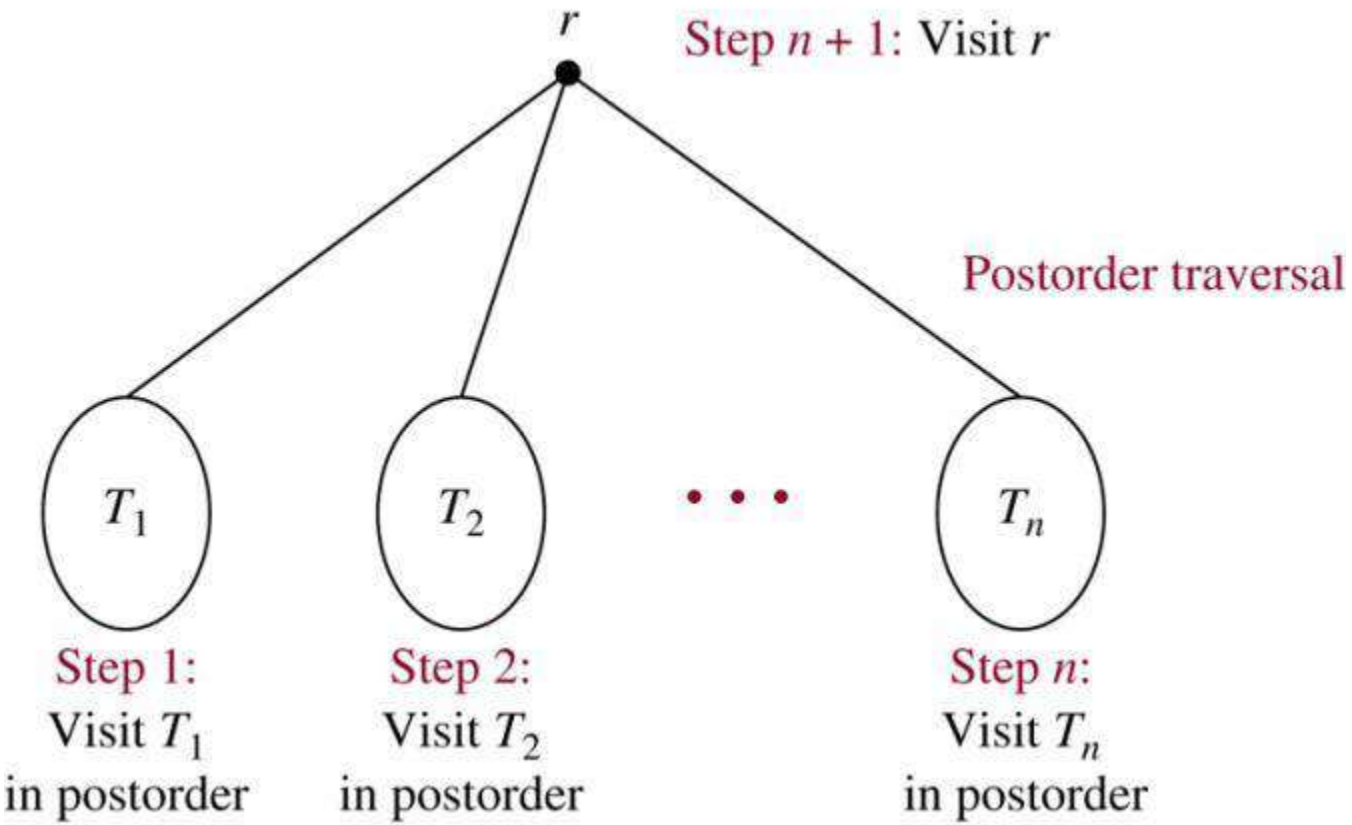
Sol:



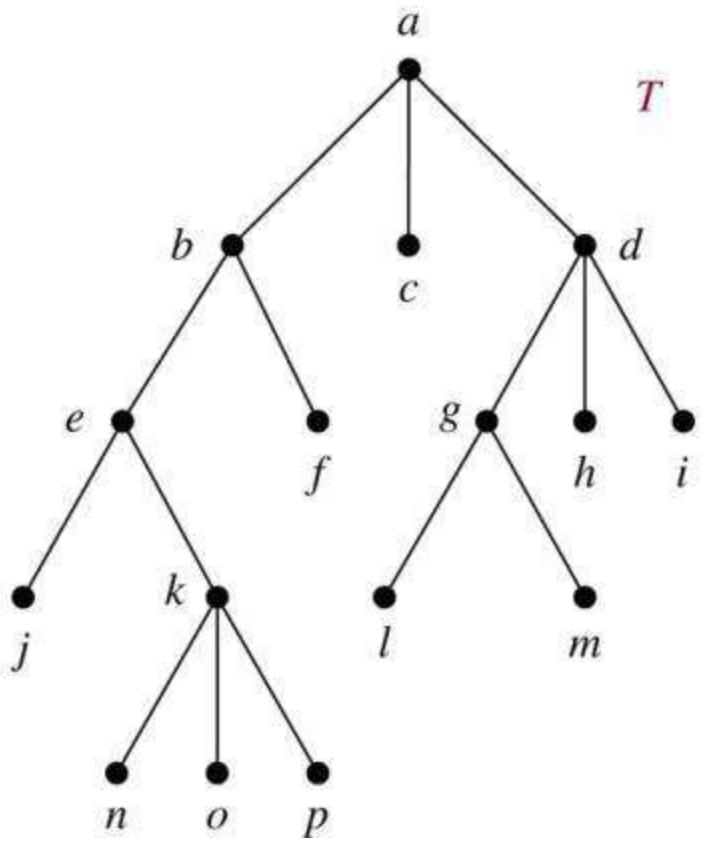
Algorithm 2 (Inorder Traversal)

Procedure *inorder*(*T*: ordered rooted tree)
r := root of *T*
If *r* is a leaf **then** list *r*
else
begin
    l := first child of *r* from left to right
    T(*l*) := subtree with *l* as its root
    inorder(*T*(*l*))
    list *r*
    for each child *c* of *r* except for *l* from left to right
        T(*c*) := subtree with *c* as its root
        inorder(*T*(*c*))
end

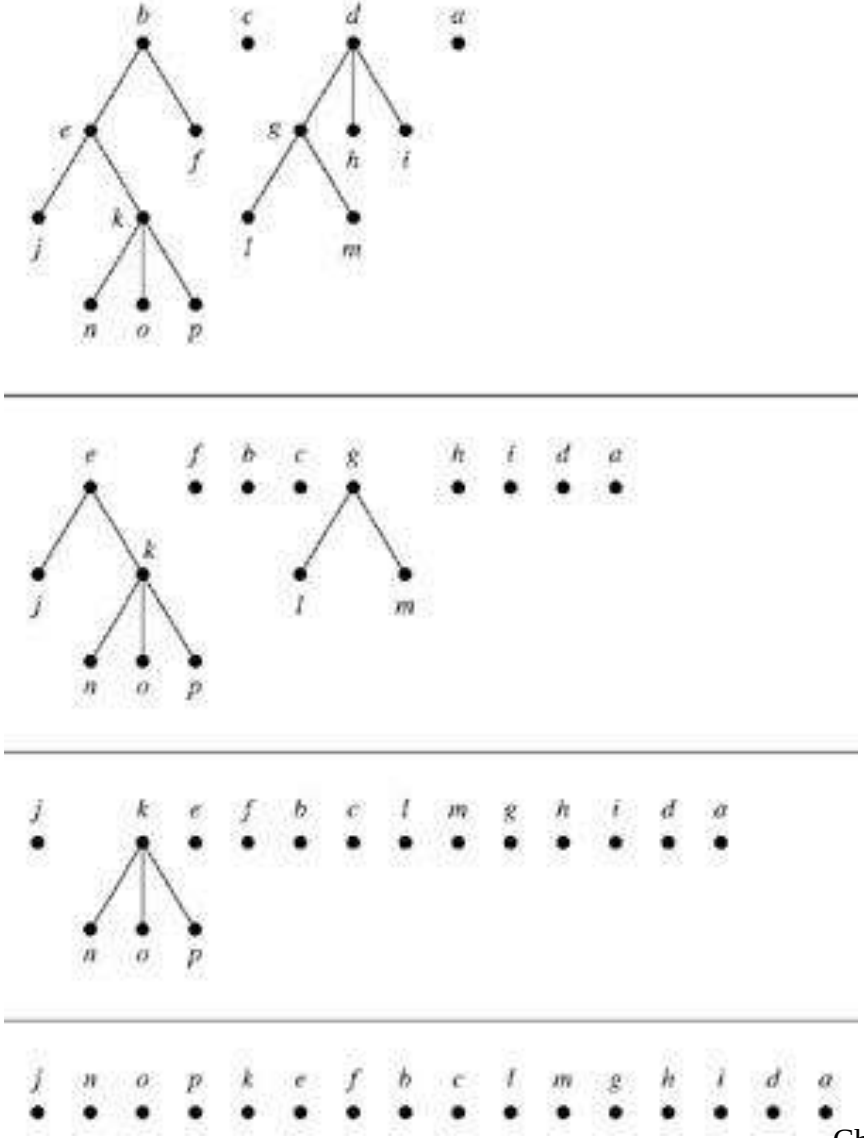
Postorder traversal(後序)



Example 4. In which order does a preorder traversal visit the vertices in the ordered rooted tree T shown below?



Sol:



Algorithm 3 (Postorder Traversal)

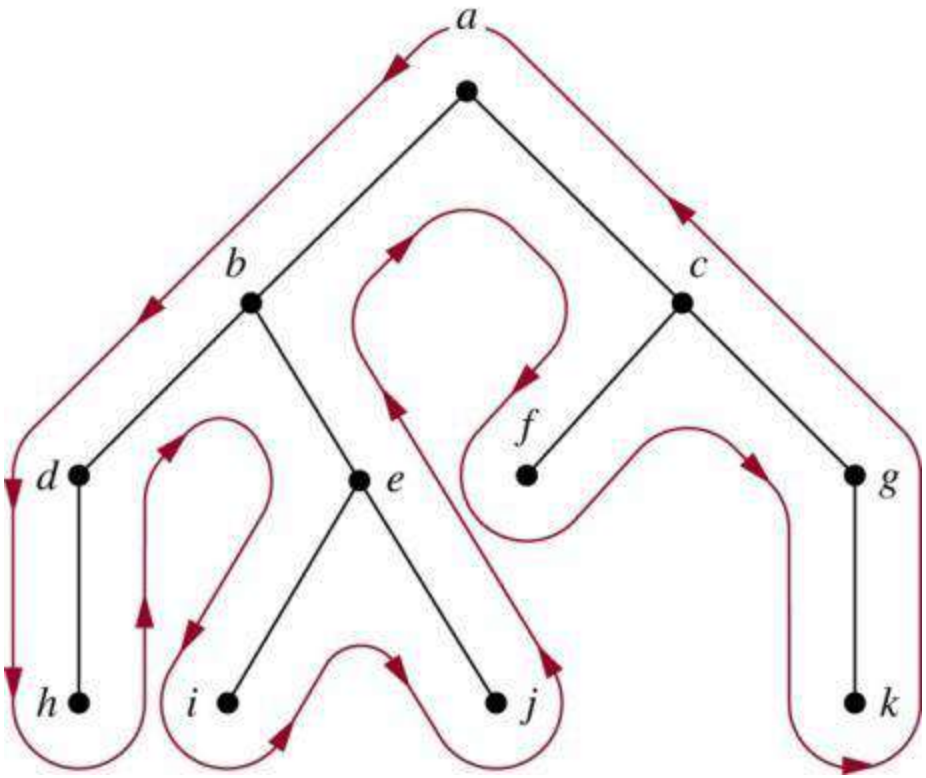
Procedure *postorder*(*T*: ordered rooted tree)
r := root of *T*
for each child *c* of *r* from left to right
begin
    T(*c*) := subtree with *c* as its root
    postorder(*T*(*c*))
end
list *r*

比較容易的表示法：依紅線的走法拜訪節點

Preorder:　curve第一次通過該點時就list該節點

Inorder:　curve第一次通過一個leaf時就list它，第二次通過一個internal節點時就list它

Postorder:　curve最後一次通過該點時就list該節點



Preorder:

$a, b, d, h, e, i, j, c, f, g, k$

Inorder:

$h, d, b, i, e, j, a, f, c, k, g$

Postorder:

$h, d, i, j, e, b, f, k, g, c, a$

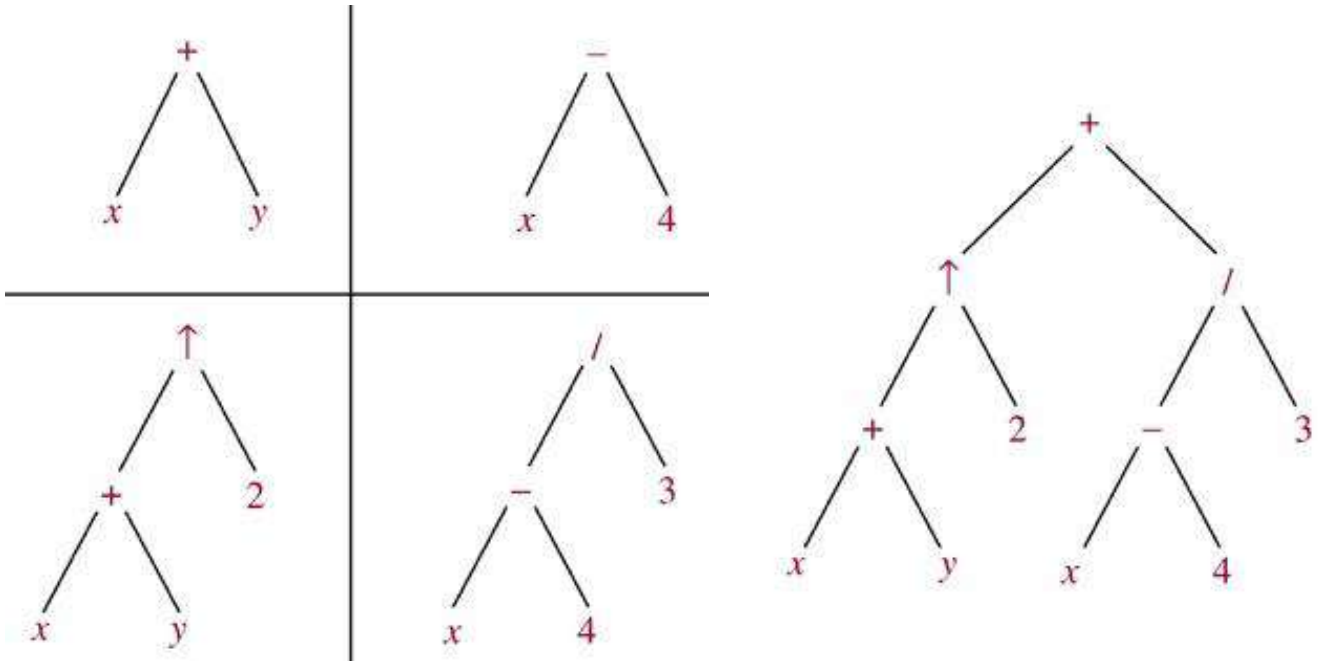
Infix, Prefix, and Postfix Notation

We can represent complicated expressions, such as compound propositions, combinations of sets, and arithmetic expressions using ordered rooted trees.

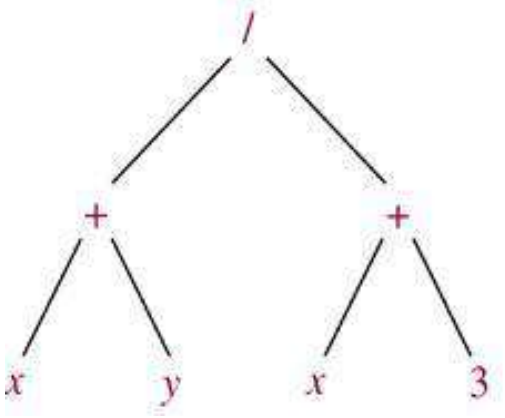
Example 1 Find the ordered rooted tree for $((x+y)\uparrow 2)+((x-4)/3)$. (↑表示次方)

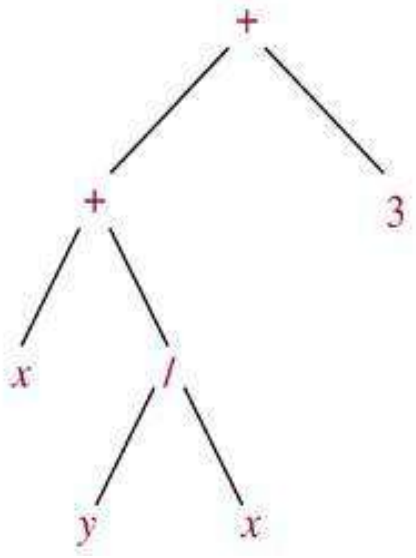
Sol.

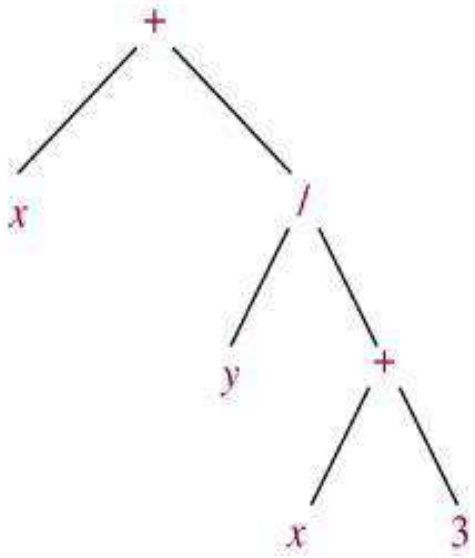
leaf:
variable
internal vertex:
operation on its left and right subtrees



The following binary trees represent the expressions:
(x+y)/(x+3), (x+(y/x))+3, x+(y/(x+3)).
All their inorder traversals lead to x+y/x+3 $\Rightarrow$ ambiguous
$\Rightarrow$ need parentheses







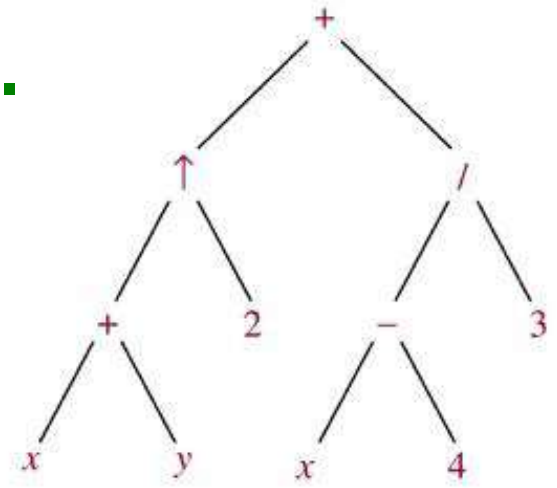
Infix form: An expression obtained when we traverse its rooted tree with inorder.

Prefix form: … … by preorder. (also named Polish notation)

Postfix form: … … by postorder. (reverse Polish notation)

Example 6 What is the prefix form for $((x+y)\uparrow 2)+((x-4)/3)$?

Sol.



$$+ \uparrow + x\ y\ 2\ /\ -\ x\ 4\ 3$$

Example 8 What is the postfix form of the expression $((x+y)\uparrow 2)+((x-4)/3)$?

Sol.

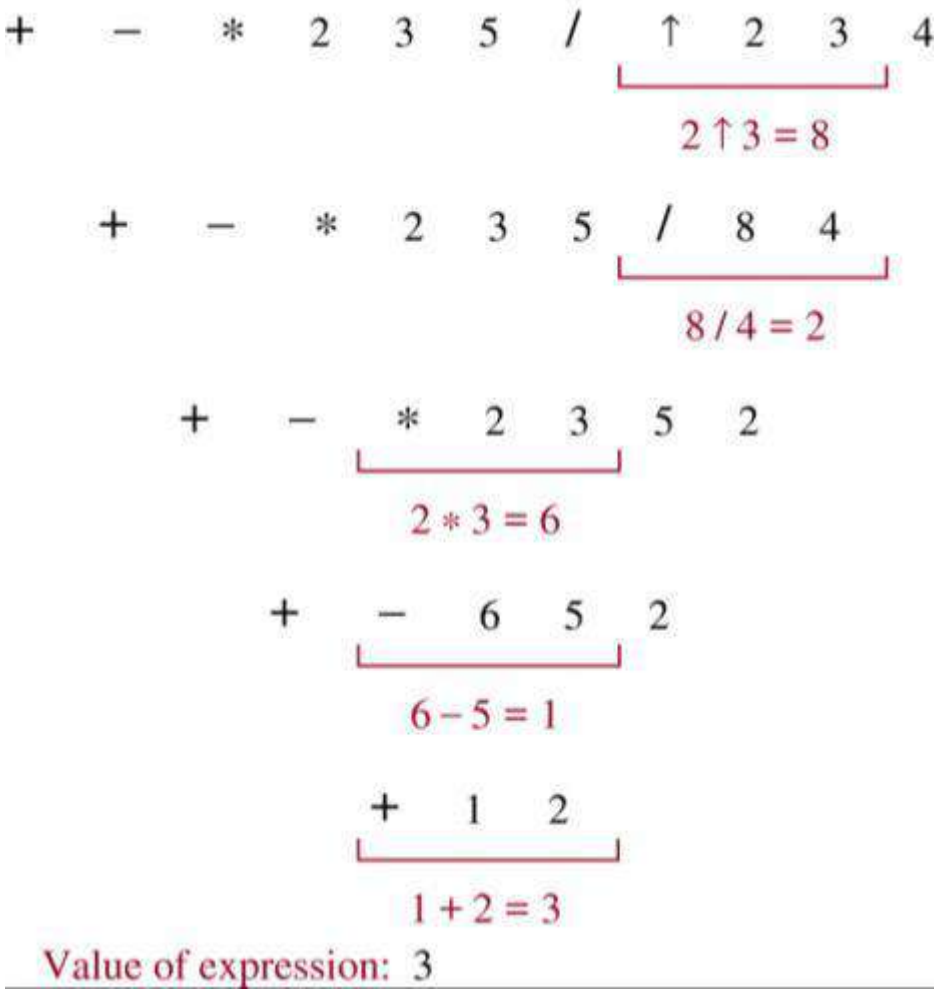
$$x\ \ y + 2 \uparrow x\ 4 - 3\ /\ +$$

Note. An expression in prefix form or postfix form is unambiguous, so no parentheses are needed.

Example 7 What is the value of the prefix expression + – * 2 3 5 / ↑ 2 3 4?

Sol.

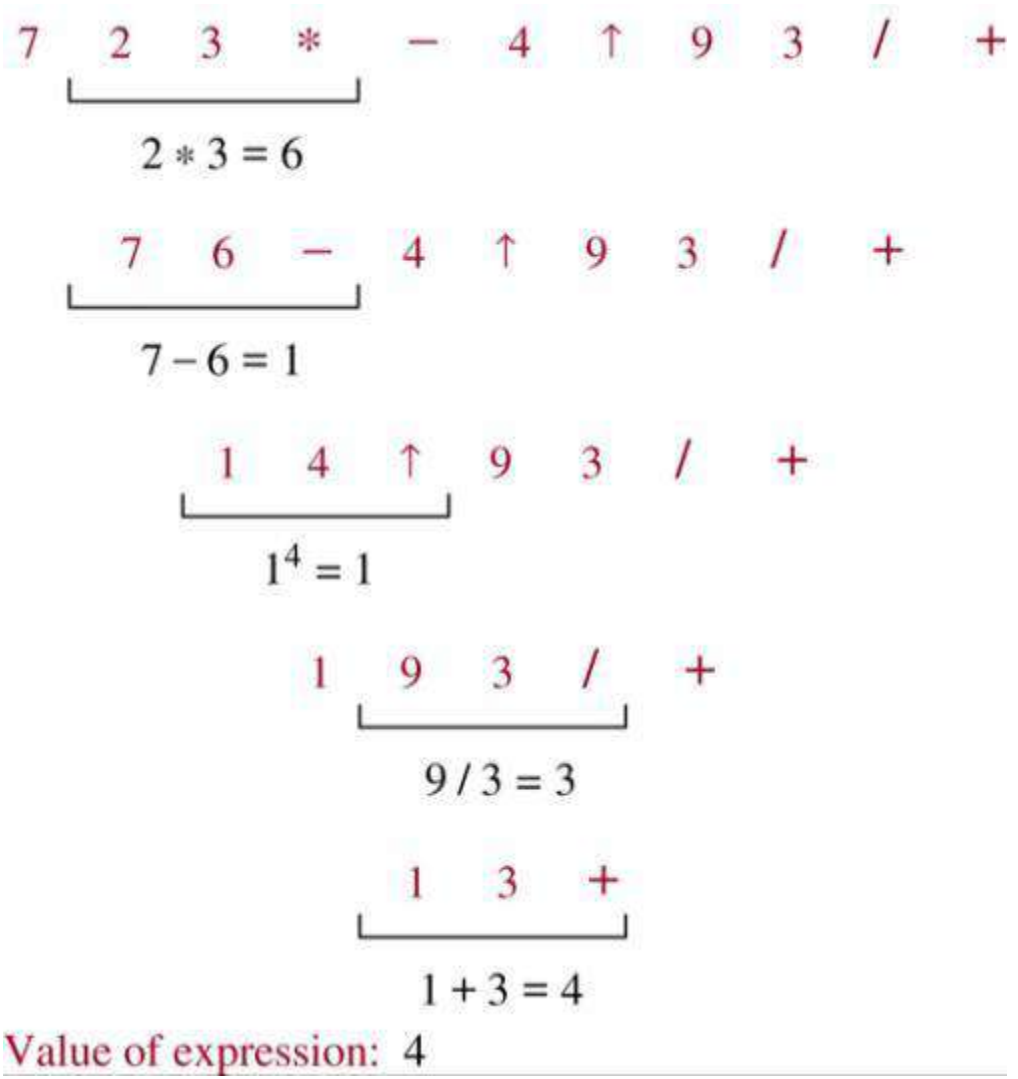
由右到左運算，將第一個出現的
運算記號(如↑)右邊的兩個數字
做此運算，運算結果取代原先位置，
依此類推。



Example 9 What is the value of the postfix expression 7 2 3 * – 4 ↑ 9 3 / +?

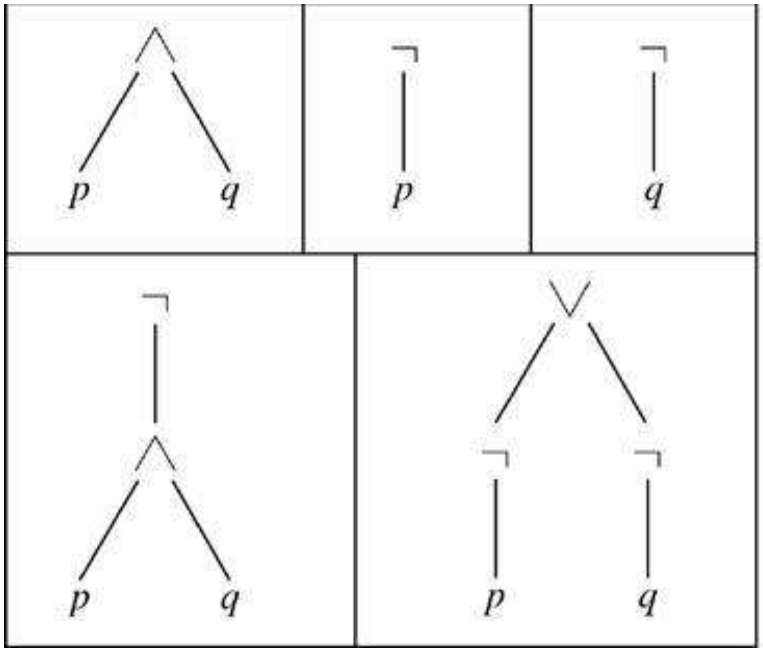
Sol.

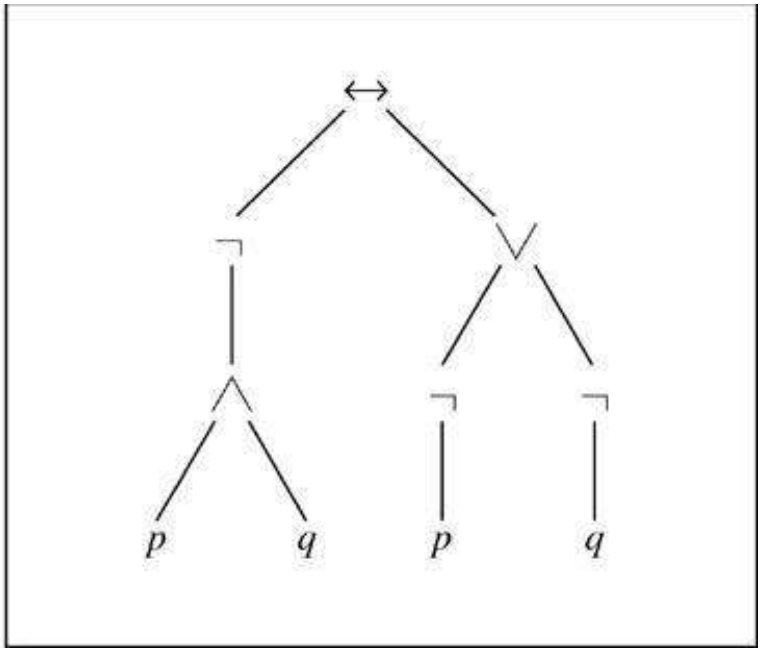
由左到右運算，將第一個出現的運算記號(如*)左邊的兩個數字做此運算，運算結果取代原先位置，依此類推。



Example 10 Find the ordered rooted tree representing the compound proposition $(\neg(p \wedge q)) \leftrightarrow (\neg p \vee \neg q)$. Then use this rooted tree to find the prefix, postfix, and infix forms of this expression.

Sol.





prefix: $\leftrightarrow \neg \wedge p\ q \vee \neg p \neg q$

postfix: $p\ q \wedge \neg p \neg q \neg \vee \leftrightarrow$

infix: $(\neg(p \wedge q)) \leftrightarrow ((\neg p) \vee (\neg q))$

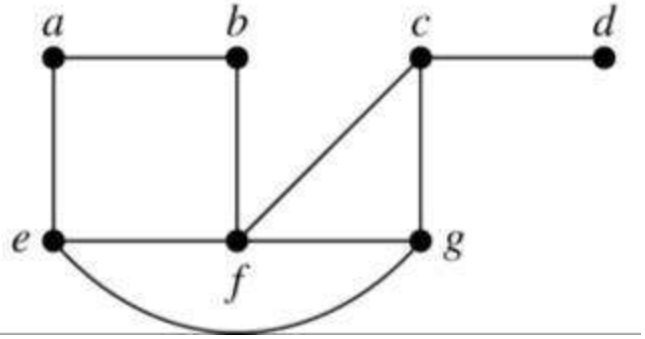
Exercise : 17, 23, 24

10.4 Spanning Trees

Introduction

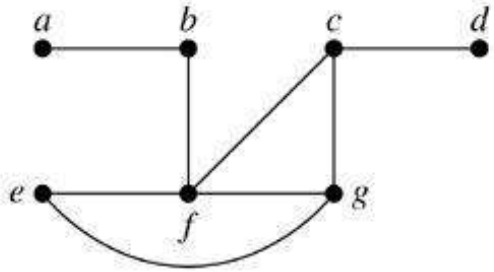
Def. Let G be a simple graph. A spanning tree of G is a subgraph of G that is a tree containing every vertex of G.

Example 1 Find a spanning tree of G.



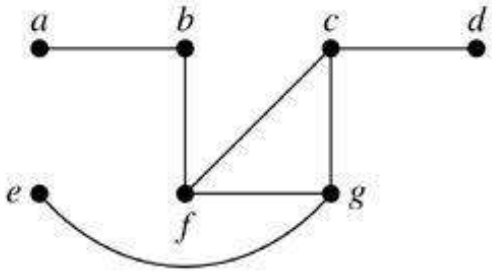
Sol.

Remove an edge from any circuit.
(repeat until no circuit exists)



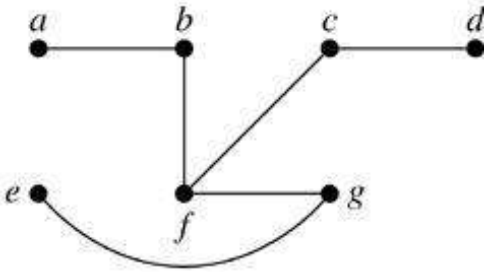
Edge removed: $\{a, e\}$

(a)



$\{e, f\}$

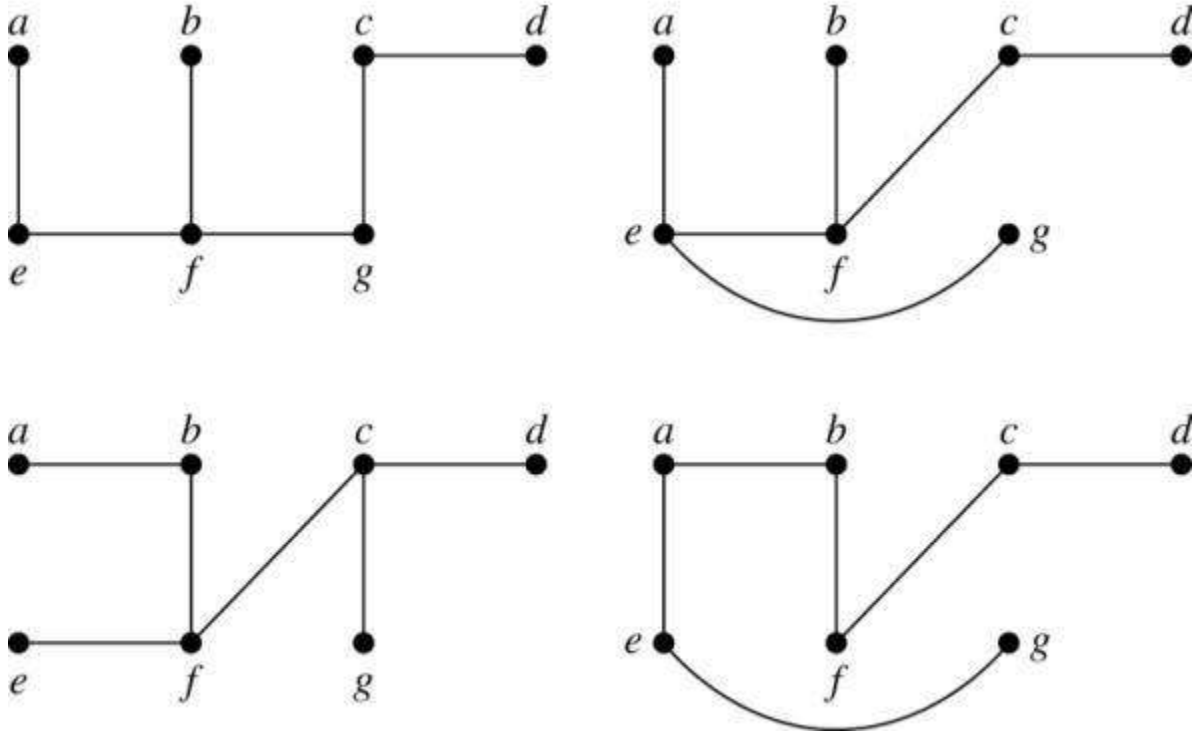
(b)



$\{c, g\}$

(c)

Four spanning trees of G:



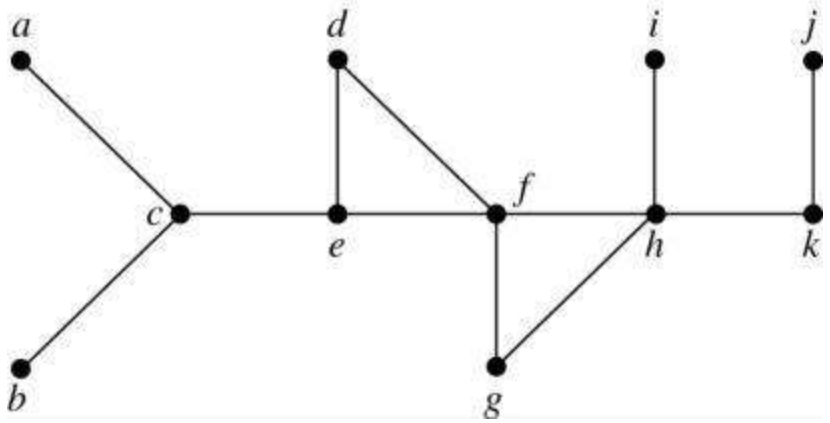
Exercise : 1, 8, 11

Thm 1 A simple graph is connected if and only if it has a spanning tree.

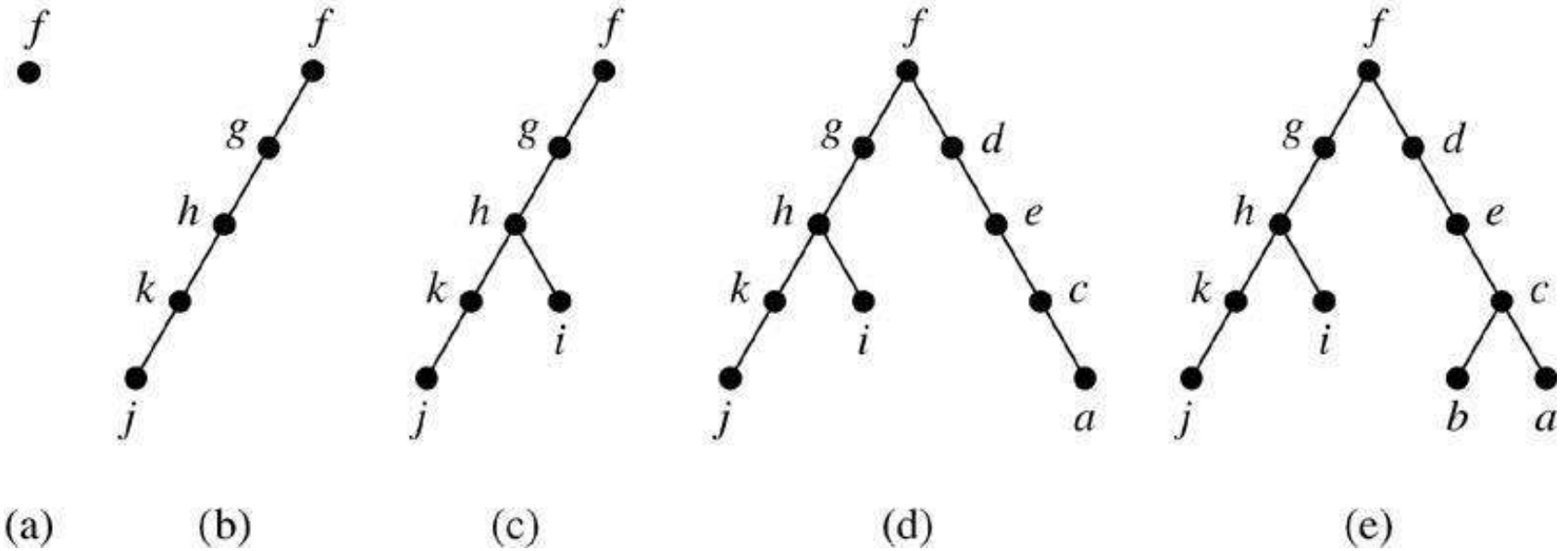
Exercise : 24, 25

Depth-First Search (DFS)

Example 3 Use depth-first search to find a spanning tree for the graph.

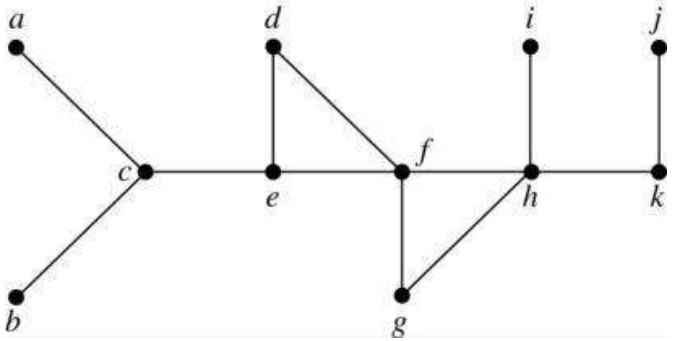


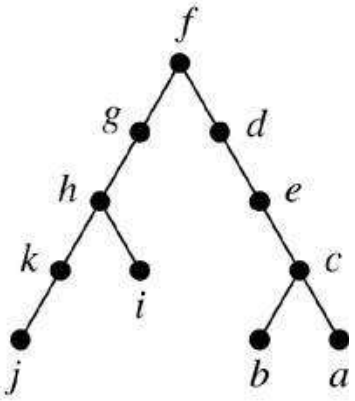
Sol. (arbitrarily start with the vertex *f*)



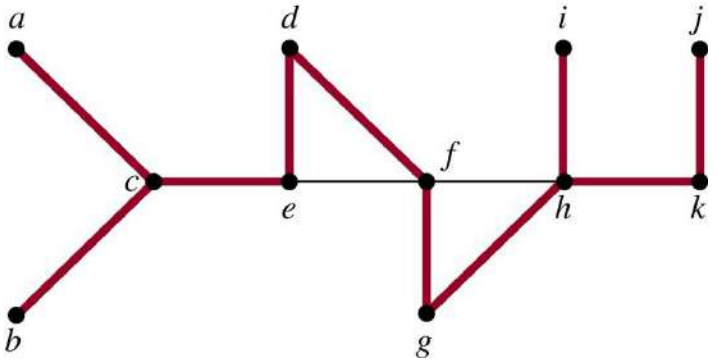
The edges selected by DFS of a graph are called tree edges. All other edges of the graph must connect a vertex to an ancestor or descendant of this vertex in the tree. These edges are called back edges.

Example 4 (承上題)





⇒



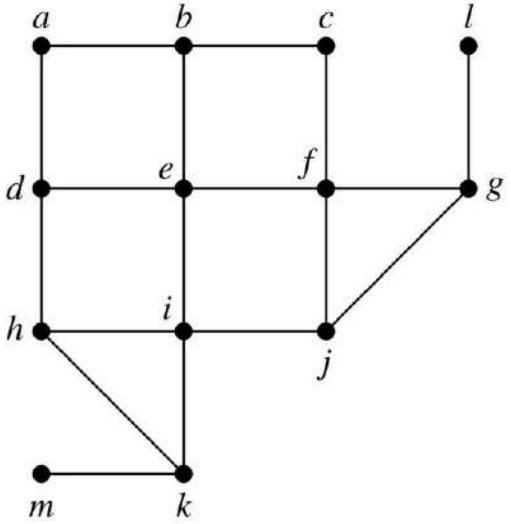
The tree edges (red) and back edges (black)

Algorithm 1 (Depth-First Search)

Procedure *DFS*(*G*: connected graph with vertices $v_1, v_2, \ldots, v_n$)
T := tree consisting only of the vertex v_1
visit(v_1)
procedure *visit*(*v*: vertex of *G*)
for each vertex *w* adjacent to *v* and not yet in *T*
begin
    add vertex *w* and edge {*v*, *w*} to *T*
    visit(*w*)
end

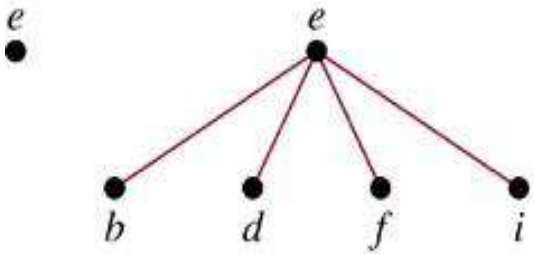
Exercise : 13

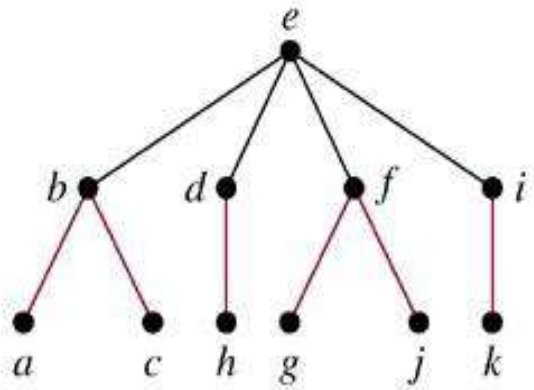
Breadth-First Search (BFS)

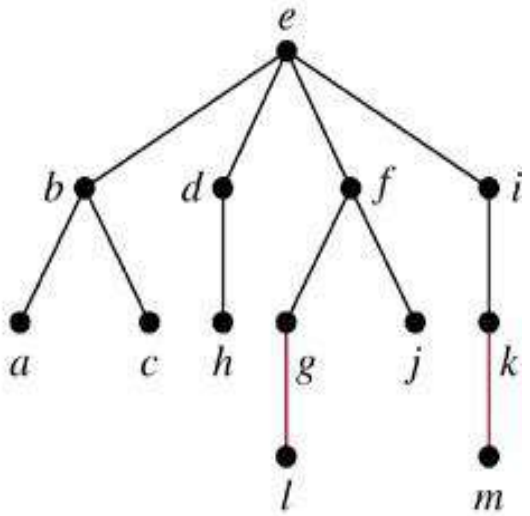


Example 5 Use breadth-first search to find a spanning tree for the graph.

Sol. (arbitrarily start with the vertex e)







Algorithm 2 (Breadth-First Search)

Procedure *BFS*(*G*: connected graph with vertices $v_1, v_2, \ldots, v_n$)
T := tree consisting only of vertex v_1
L := empty list
put v_1 in the list *L* of unprocessed vertices
while *L* is not empty
begin
    remove the first vertex *v* from *L*
    for each neighbor *w* of *v*
        if *w* is not in *L* and not in *T* **then**
        begin
            add *w* to the end of the list *L*
            add *w* and edge {*v*, *w*} to *T*
        end
end

Exercise : 16

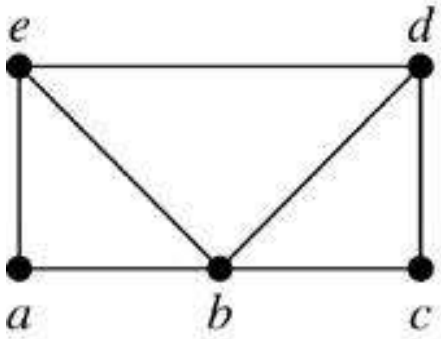
Backtracking Applications

There are problems that can be solved only by performing an exhaustive (徹底的) search of all possible solutions.

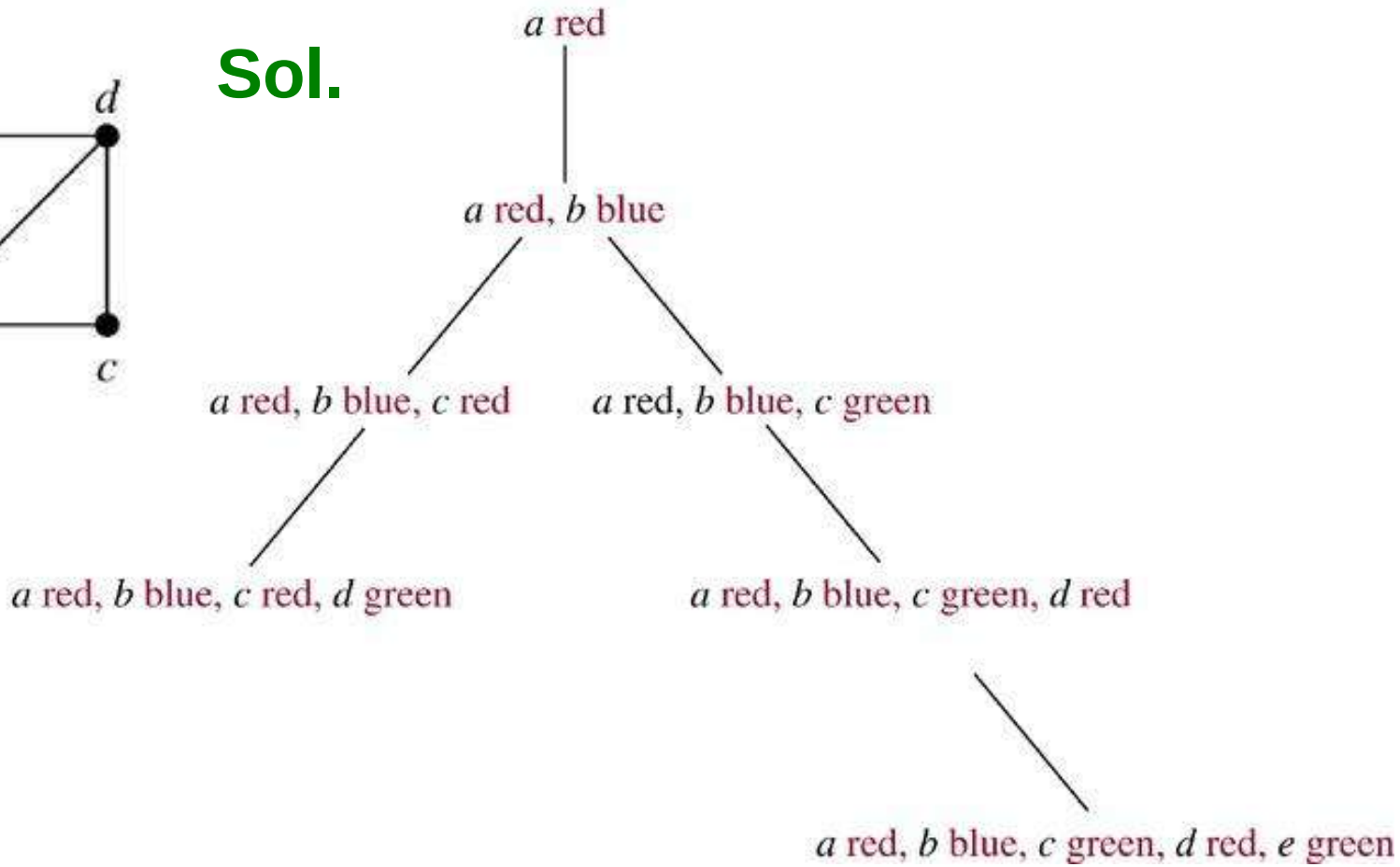
Decision tree: each internal vertex represents a decision, and each leaf is a possible solution.

To find a solution via backtracking: 在 decision tree上由root做一連串的decision走到leaf，若leaf不是solution，或整個子樹檢查完未找到解，則退到上層parent，改找另一個子樹。

Example 6 (Graph Colorings) How can backtracking be used to decide whether the following graph can be colored using 3 colors?



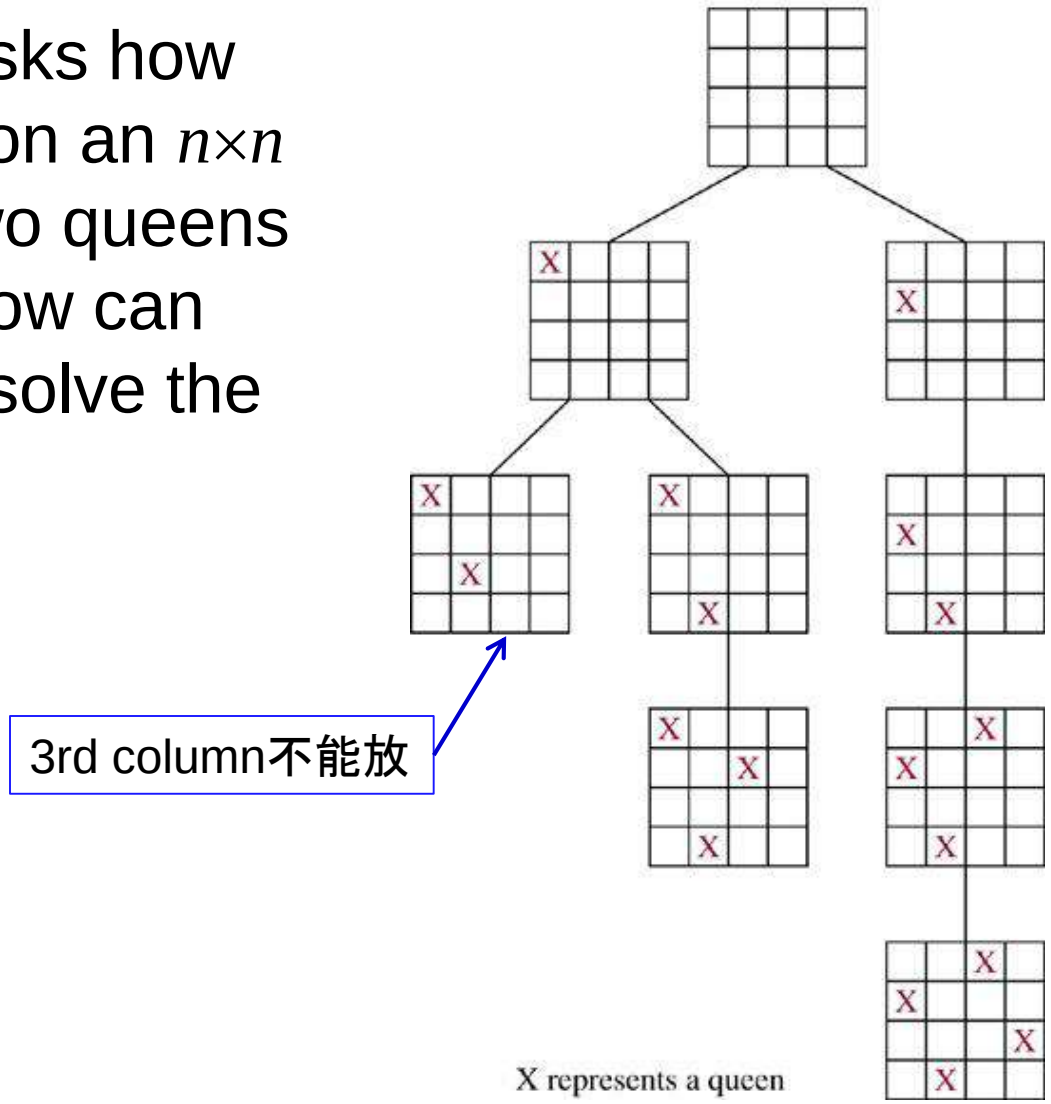
Sol.



Example 7

(The n-Queens Problem)
The n-queens problem asks how n queens can be placed on an $n \times n$ chessboard so that no two queens can attack on another. How can backtracking be used to solve the n-queens problem.

Sol. 以n=4為例



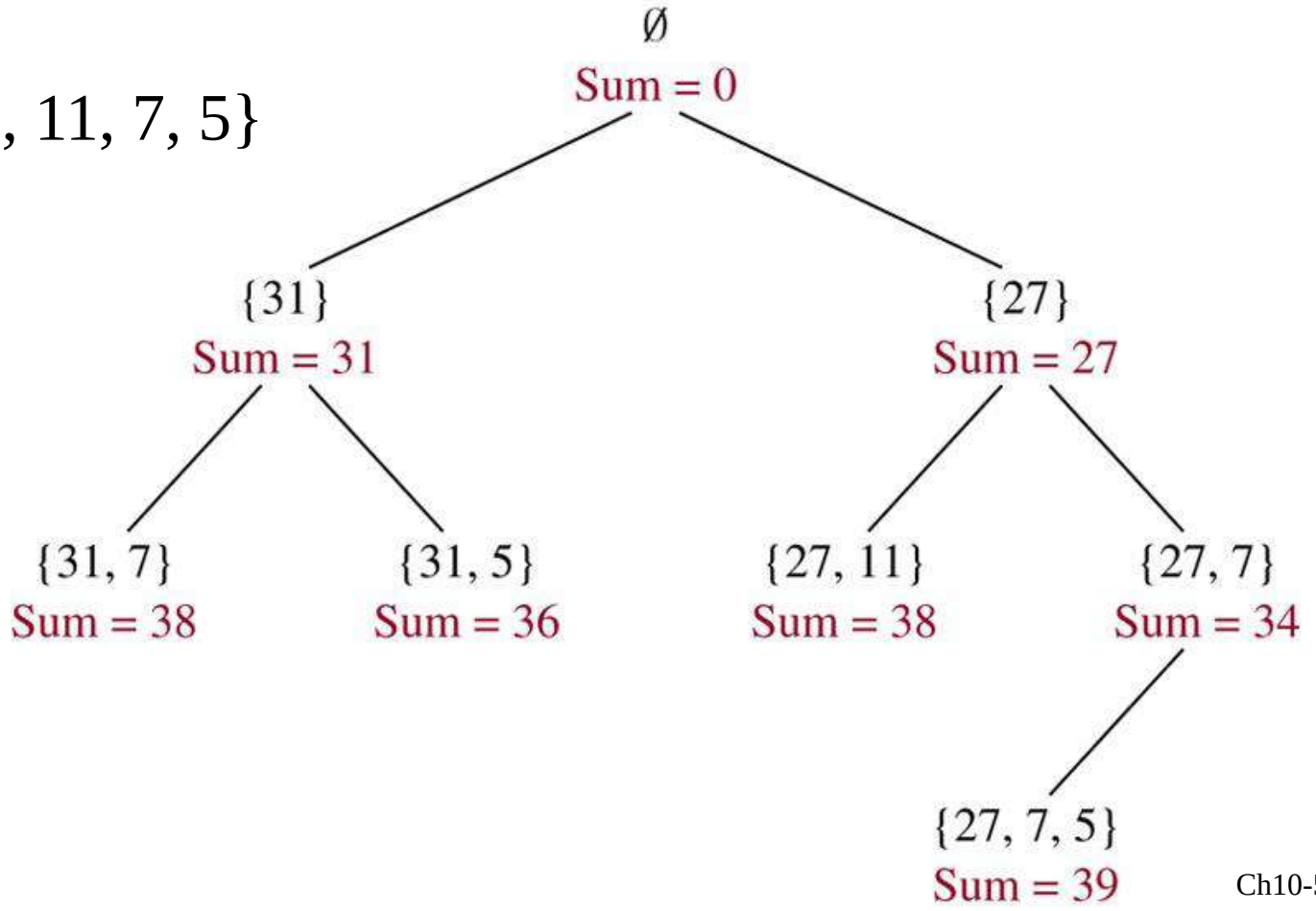
Example 8 (Sum of Subsets)

Give a set S of positive integers $x_1, x_2, \ldots, x_n$, find a subset of S that has M as its sum. How can backtracking be used to solve this problem.

Sol.

$S = \{31, 27, 15, 11, 7, 5\}$

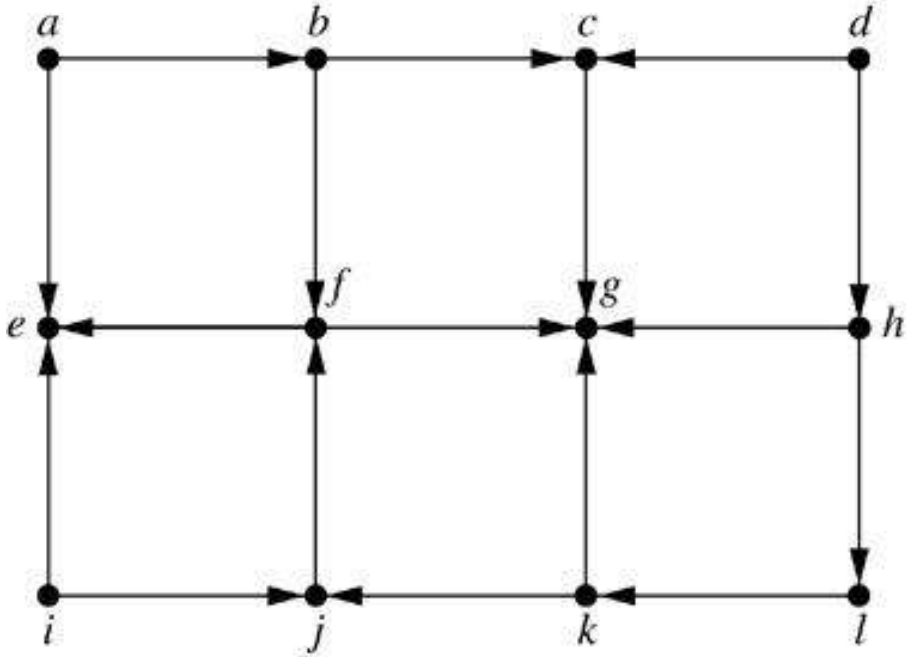
$M = 39$



Exercise : 30

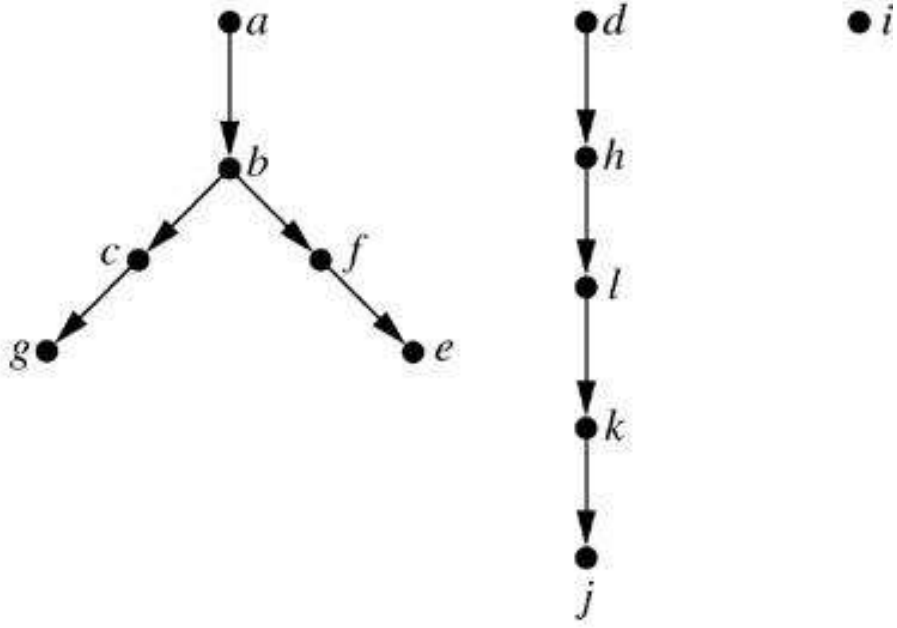
Depth-First Search in Directed Graphs

Example 9 What is the output of DFS given the graph G?



(a)

Sol.



(b)

10.5 Minimum Spanning Trees

G: connected weighted graph (each edge has an weight ≥ 0)

Def. minimum spanning tree of G: a spanning tree of G with smallest sum of weights of its edges.

Algorithms for Minimum Spanning Trees

Algorithm 1 (Prim's Algorithm)

Procedure *Prim*(G: connected weighted undirected graph with n vertices)

T := a minimum-weight edge

for i := 1 **to** n–2

begin

  e := an edge of minimum weight incident to a vertex in T and not forming a simple circuit in T if added to T

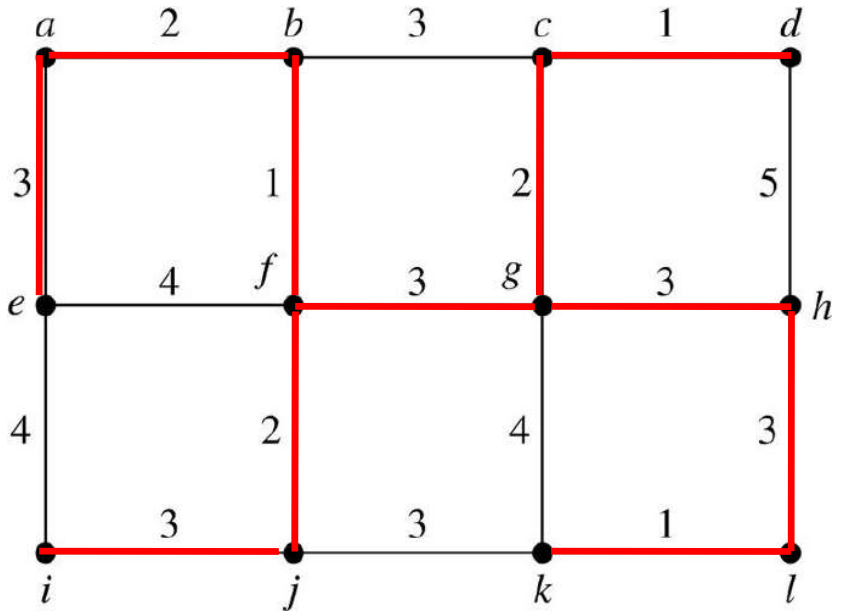
  T := T with e added

end {T is a minimum spanning tree of G}

Example 2 Use Prim's algorithm to find a minimum spanning tree of G.

Sol.

Choice	Edge	Weight
1	$\{b, f\}$	1
2	$\{a, b\}$	2
3	$\{f, j\}$	2
4	$\{a, e\}$	3
5	$\{i, j\}$	3
6	$\{f, g\}$	3
7	$\{c, g\}$	2
8	$\{c, d\}$	1
9	$\{g, h\}$	3
10	$\{h, l\}$	3
11	$\{k, l\}$	1
	Total:	24



(過程中維持只有一個tree)

Exercise: 3

Algorithm 2 (Kruskal Algorithm)

Procedure *Kruskal*(*G*: connected weighted undirected graph with *n* vertices)

T := empty graph

for *i* := 1 **to** *n*–1

begin

e := any edge in *G* with smallest weight that does not form a simple circuit when added to *T*

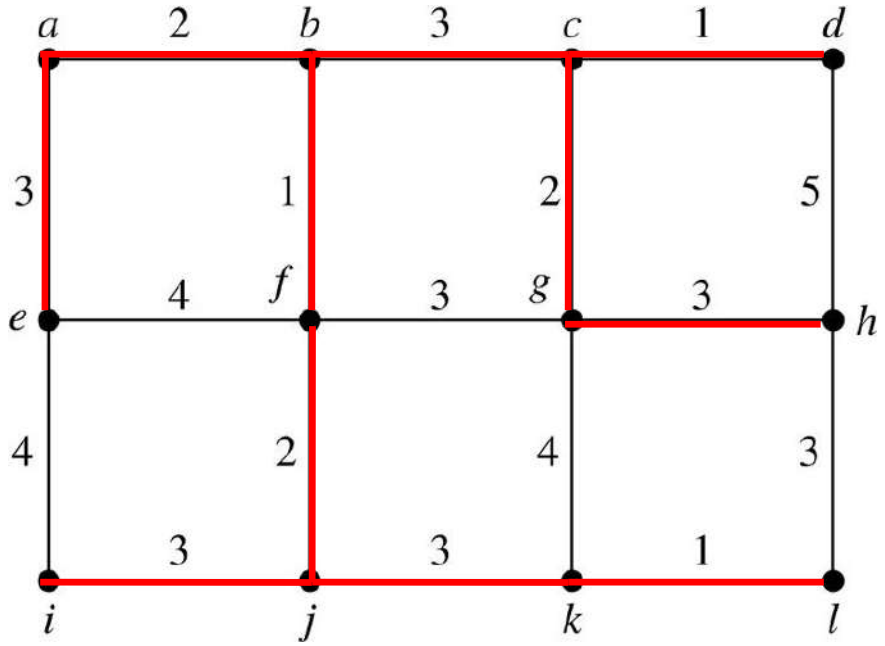
T := *T* with *e* added

end {*T* is a minimum spanning tree of *G*}

Example 3 Use Kruskal algorithm to find a minimum spanning tree of G.

Sol.

Choice	Edge	Weight
1	$\{c, d\}$	1
2	$\{k, l\}$	1
3	$\{b, f\}$	1
4	$\{c, g\}$	2
5	$\{a, b\}$	2
6	$\{f, j\}$	2
7	$\{b, c\}$	3
8	$\{j, k\}$	3
9	$\{g, h\}$	3
10	$\{i, j\}$	3
11	$\{a, e\}$	3
	Total:	24



(過程中tree通常會有好幾個)

Exercise: 7